5.1 Inverse Functions

INVERSE RELATION

Interchanging the first and second coordinates of each ordered pair in a relation produces the **inverse relation**.

$$h = \{(-8, 5), (4, -2), (-7, 1), (3.8, 6.2)\}$$

$$\text{Inverse of } h = \{(5, -8), (-2, 4), (1, -7), (6.2, 3.8)\}$$

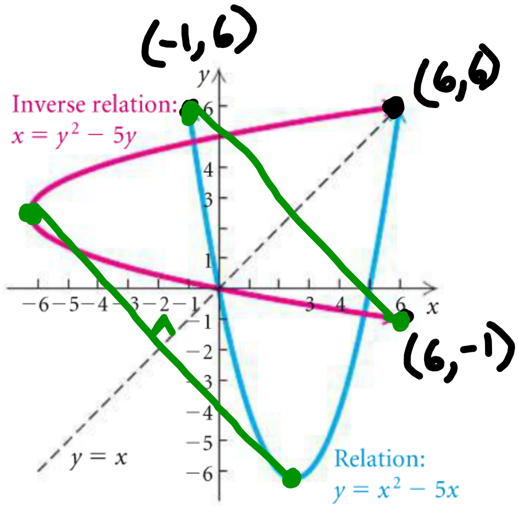
$y = x^2 - 5x.$ relation

$x = y^2 - 5y.$ inverse relation

If a relation is given by an equation, then the solutions of the inverse can be found from those of the original equation by interchanging the first and second coordinates of each ordered pair. Thus the graphs of a relation and its inverse are always reflections of each other across the line $y = x$.

(the identity function)

$x = y^2 - 5y$	y
6	−1
0	0
−6	2
−4	4
6	6



x	$y = x^2 - 5x$
−1	6
0	0
2	−6
4	−4
6	6

Recall, if a relation is a function then the x-coordinates do **not** repeat; that is, it passes the vertical line test.

Find the inverse of the relation.

1. $\{(7,8),(-2,8),(3,-4),(8,-8)\}$

$\{(8,7),(8,-2),(-4,3),(-8,8)\}$ ← not a function

Find an equation of the inverse relation.

5. $y = 4x - 5$

$x = 4y - 5$

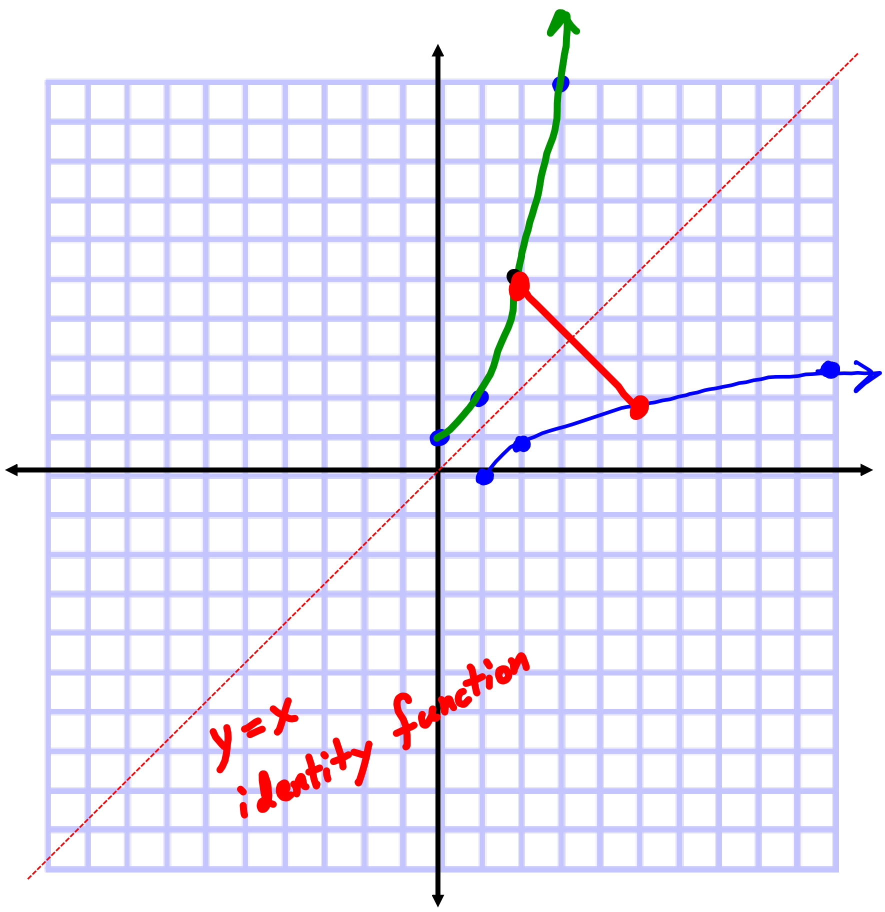
7. $x^3y = -5$ $y^3x = -5$

Graph the equation by substituting and plotting points. Then reflect the graph across the line $y = x$ to obtain the graph of its inverse.

12. $y = x^2 + 1$, $x \geq 0$

x	y
0	1
1	2
2	5
3	10

y	x
1	0
2	1
5	2
10	3



HORIZONTAL-LINE TEST

If every possible horizontal line intersects a function f(x) one time at most, then f(x) is a one-to-one function. That is, it passes the **horizontal line test**.

A function f(x) has an inverse function, $f^{-1}(x)$, if and only if f(x) is one-to-one. That is, its inverse image is also a function.

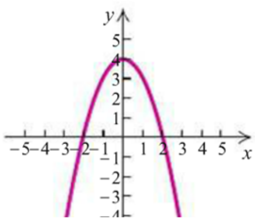
If the inverse of a function f is also a function, it is named f^{-1} (read "f-inverse").

The -1 in f^{-1} is *not* an exponent!

Do *not* misinterpret the -1 in f^{-1} as a negative exponent: f^{-1} does *not* mean the reciprocal of f and $f^{-1}(x)$ is *not* equal to $\dfrac{1}{f(x)}$.

Using the horizontal-line test, determine whether the function is one-to-one. Does the inverse function exist?

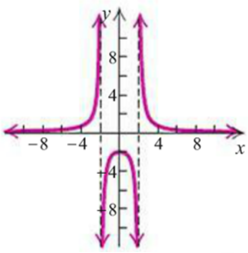
27. $f(x) = 4 - x^2$



not one-to-one

f^{-1} does not exist

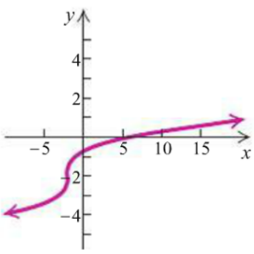
29. $f(x) = \dfrac{8}{x^2 - 4}$



not one-to-one

f^{-1} does not exist

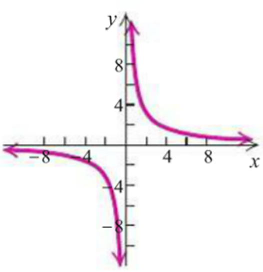
31. $f(x) = \sqrt[3]{x+2} - 2$



one-to-one

f^{-1} exists

32. $f(x) = \dfrac{8}{x}$



one-to-one

f^{-1} exists

If a function f is one-to-one, then f^{-1} is the unique function such that each of the following holds:

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x,$ for each x in the domain of f, and
$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x,$ for each x in the domain of f^{-1}.

For the function f, use composition of functions to show that f^{-1} is as given.

84. $f(x) = \dfrac{x + 5}{4}, \quad f^{-1}(x) = 4x - 5$

$$(f \circ f^{-1})(x) = f(4x-5) = \frac{(4x-5)+5}{4} = \frac{4x}{4} = x$$

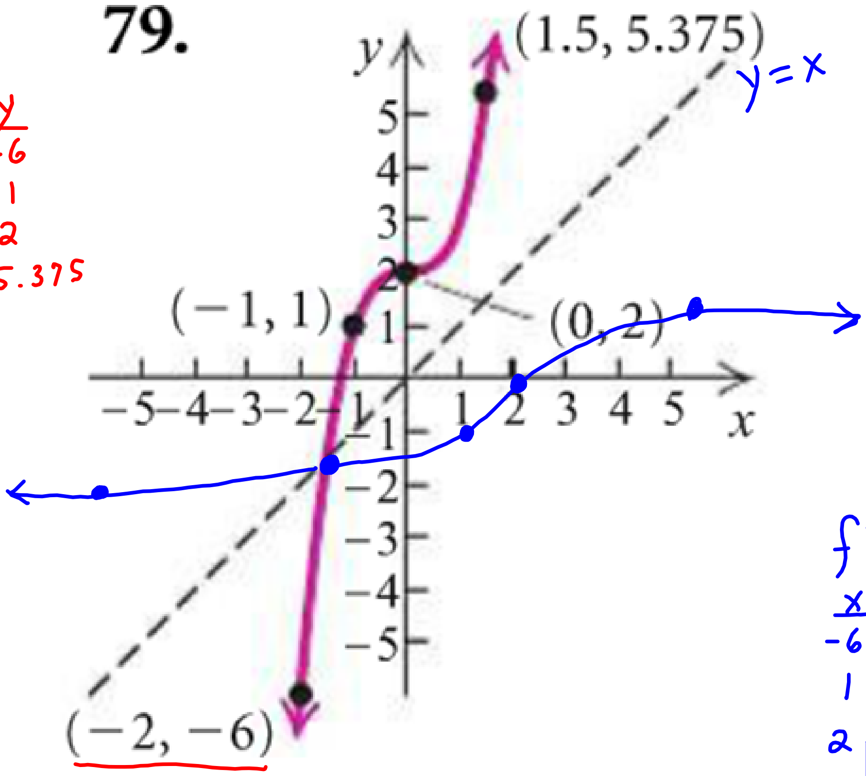
$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{x+5}{4}\right) = 4\left(\frac{x+5}{4}\right) - 5 = x+5-5 = x$$

Sketch the graph of the inverse function f^{-1}.

f

x	y
-2	-6
-1	1
0	2
1.5	5.375

79.



f^{-1}

x	y
-6	-2
1	-1
2	0
5.375	1.5

Finding an inverse informally

(1) $f(x) = 8x$

$f^{-1}(x) = \frac{x}{8}$

formal approach

$y = 8x \rightleftarrows$

$x = 8y$

$y = \frac{x}{8}$

(2) $g(x) = x - 3$

$g^{-1}(x) = x + 3$

$y = x - 3 \rightleftarrows$

$x = y - 3$

$y = x + 3$

(3) $h(x) = 9x + 2$

a) $\frac{x-2}{9}$

b) $\frac{x}{9} - 2$

$y = 9x + 2 \rightleftarrows$

$x = 9y + 2$

$x - 2 = 9y$

$\frac{x-2}{9} = y$