5.4 Properties of Logarithmic Functions

THE PRODUCT RULE 1

For any positive numbers M and N and any logarithmic base a,

$$\log_a MN = \log_a M + \log_a N.$$

(The logarithm of a product is the sum of the logarithms of the factors.)

Exponent Rule

$a^x a^y = a^{x+y}$

Given $243 = 9 \cdot 27$, Find $\log_3 243$

$\log_3 243 = \log_3 9 + \log_3 27$

$= 2 + 3 = 5.$

Check: $3^5 = 243$

Proof of 1 + 2

if $M = a^x$
then $\log_a M = x$

if $N = a^y$
then $\log_a N = y$

Let $M = a^x$ and $N = a^y$

Then $M/N = a^x / a^y \Rightarrow$

$M/N = a^{x-y} \Rightarrow$

$\log_a M/N = x - y$

$\log_a M/N = \log_a M - \log_a N$

QED

Express as a sum of logarithms.

4. $\log_4 (64 \cdot 4) =$

$\log_4 64 + \log_4 4$

$3 + 1 = 4$

8. $\ln ab =$

$\ln a + \ln b$

THE QUOTIENT RULE 2

For any positive numbers M and N, and any logarithmic base a,

$$\log_a \frac{M}{N} = \log_a M - \log_a N.$$

(The logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.)

$$\frac{a^n}{a^m} = a^{n-m}$$

Express as a difference of logarithms.

18. $\log_a \frac{76}{13} = \log_a 76 - \log_a 13$

21. $\ln \frac{r}{s} = \ln r - \ln s$

$2^0 = 1$

★) $\log_2 \frac{1}{8} = \log_2 1 - \log_2 8$

$= 0 - 3 = -3$

THE POWER RULE 3

For any positive number M, any logarithmic base a, and any real number p,

$$\log_a M^p = p \log_a M.$$

(The logarithm of a power of M is the exponent times the logarithm of M.)

$(a^n)^m = a^{nm}$

Express as a product.

$a^{\frac{n}{m}} = \sqrt[m]{a^n}$

12. $\ln y^5 = 5 \ln y$

15. $\ln \sqrt[3]{4^1} = \ln 4^{\frac{1}{3}} = \frac{1}{3} \ln 4$

16. $\ln \sqrt[2]{a^1} = \ln a^{\frac{1}{2}} = \frac{1}{2} \ln a$

$10^{?} = 10^{3/2}$

★) $\log_{10} \sqrt{1000} = \log 1000^{\frac{1}{2}}$

$= \log (10^3)^{\frac{1}{2}}$

$= \log_{10} 10^{3/2} = 3/2$

Common Errors

$\log_a MN \neq (\log_a M)(\log_a N)$

$\log_a (M+N) \neq \log_a M + \log_a N$

$\log_a \frac{M}{N} \neq \frac{\log_a M}{\log_a N}$

$(\log_a M)^p \neq p \log_a M$

① $\log_a M + \log_a N$
this is not the distrib

② $\log_a M - \log_a N$

③ $\log_a M^p = p \log_a M$

Express in terms of sums and differences of logarithms.

$$\log \frac{ab^2t^4}{ck^3} = \log a + \log b^2 + \log t^4 - \log c - \log k^3$$
$$= \log a + 2\log b + 4\log t - \log c - 3\log k$$

26. $\log_b \frac{x^2y}{b^3} = \log_b x^2 + \log_b y - \log_b b^3$

$= 2\log_b x + \log_b y - 3$

30. $\ln \sqrt[3]{5x^5} = \ln (5x^5)^{1/3}$

$= \ln (5^{\frac{1}{3}} x^{5/3})$

$= \ln 5^{\frac{1}{3}} + \ln x^{5/3}$

$= \frac{1}{3}\ln 5 + \frac{5}{3}\ln x$

32. $\log_c \sqrt[3]{\dfrac{y^3 z^2}{x^4}}$

$$= \log_c \left(\frac{y^3 z^2}{x^4}\right)^{\frac{1}{3}} = \log_c \frac{y\, z^{2/3}}{x^{4/3}}$$

$$= \log_c y + \log_c z^{2/3} - \log_c x^{4/3}$$

$$= \log_c y + \frac{2}{3}\log_c z - \frac{4}{3}\log_c x$$

34. $\log_a \sqrt{\dfrac{a^6 b^8}{a^2 b^5}}$

$$= \log_a \left(a^4 b^3\right)^{\frac{1}{2}}$$

$$= \log_a \left(a^2 b^{3/2}\right)$$

$$= \log_a a^2 + \log_a b^{3/2}$$

$$= 2 + \frac{3}{2}\log_a b$$

Express as a single logarithm and, if possible, simplify.

$$\ln x - 3\ln y + \frac{1}{2}\ln w - 5\ln k - \frac{1}{3}\ln B + \ln A$$

$$\ln\left(\frac{x \cdot w^{1/2} \cdot A}{y^3 \cdot k^5 \cdot B^{1/3}}\right) = \ln \frac{A\sqrt{w}\; x}{\sqrt[3]{B}\, k^5 y^3}$$

40. $\frac{1}{2}\log a - \log 2$ $= \log a^{\frac{1}{2}} - \log 2$

$= \log \frac{\sqrt{a}}{2}$

41. $\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x$ $=$

$\log_a x^{\frac{1}{2}} + \log_a y^4 - \log_a x^3 =$

$\log_a \frac{x^{\frac{1}{2}} y^4}{x^3} = \log_a x^{(\frac{1}{2}-3)} y^4$ $\quad -\frac{6}{2}$

$= \log_a x^{-5/2} y^4 = \log_a \frac{y^4}{x^{5/2}}$

$\frac{a^{-2}b^3c^{-4}}{d^{-1}e^2f^{-2}} =$

$\frac{b^3 d f^2}{a^2 c^4 e^2}$

42. $\frac{2}{5}\log_a x - \frac{1}{3}\log_a y$ $=$

$\log_a x^{2/5} - \log_a y^{\frac{1}{3}} = \log_a \frac{x^{2/5}}{y^{1/3}}$

47. $\log(x^2 - 5x - 14) - \log(x^2 - 4)$ $=$

$\log_{10} \frac{x^2 - 5x - 14}{x^2 - 4} = \log \frac{(x-7)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}$

$= \log \frac{x-7}{x-2}$

Given that $\log_a 2 \approx 0.606$, $\log_a 7 \approx 1.700$, *and* $\log_a 11 \approx 2.095$, *find each of the following, if possible. Round the answer to the nearest thousandth.*

55. $\log_a(98)$

$$= \log_a(2 \cdot 49)$$
$$= \log_a 2 + \log_a 49$$
$$= 0.606 + \log_a 7^2$$
$$= 0.606 + 2\log_a 7$$
$$= 0.606 + 2(1.700)$$
$$= \begin{array}{r} 3.400 \\ .606 \\ \hline 4.006 \end{array}$$

THE LOGARITHM OF A BASE TO A POWER

For any base a and any real number x,

$$\log_a a^x = x.$$

(The logarithm, base a, of a to a power is the power.)

A BASE TO A LOGARITHMIC POWER

For any base a and any positive real number x,

$$a^{\log_a x} = x.$$

(The number a raised to the power $\log_a x$ is x.)

$f(x) = \log_b x$

$f^{-1}(x) = b^x$

$(f \circ f^{-1})(x) =$

$\log_b b^x = x$

$(f^{-1} \circ f)(x) =$

$b^{\log_b x} = x$

66. $\log_t t^{2713} = 2713$

69. $3^{\log_3 4x} = 4x$

70. $5^{\log_5 (4x-3)} = 4x-3$

71. $10^{\log w} = w$

72. $e^{\ln x^3} = x^3$

74. $\log 10^{-k} = -k$

75. $\log_b \sqrt{b} = \log_b b^{\frac{1}{2}} = \frac{1}{2}$