COMPARING THREE OR MORE MEANS (ONE-WAY ANALYSIS OF VARIANCE)

Definition

Analysis of Variance (ANOVA) is an inferential method used to test the equality of three or more population means.

In Other Words

In ANOVA, the null hypothesis is always that the means of the different populations are equal. The alternative hypothesis is always that at least one population mean is different from the others.

It is tempting to test the null hypothesis $H_0: \mu_1 = \mu_2 = \mu_3$ by comparing

$$H_0: \mu_1 = \mu_2 \quad \text{and} \quad H_0: \mu_1 = \mu_3 \quad \text{and} \quad H_0: \mu_2 = \mu_3$$
$$H_1: \mu_1 \neq \mu_2 \qquad\qquad H_1: \mu_1 \neq \mu_3 \qquad\qquad H_1: \mu_2 \neq \mu_3$$

Each test would have a probability of a Tvpe I error (rejecting the null hypothesis when it is true) of α. If

*In general, there would be $\binom{k}{2} = {}_kC_2$ pairs to test, where k equals the number of population means to be compared.

For example, a family doctor might wonder if the mean HDL (so-called good) levels of cholesterol of males in the age groups 20 to 29, 40 to 49, and 60 to 69 years old is the same or different. To test this, we assume that the mean HDL cholesterol of each age group is the same (remember, the null hypothesis is always a statement of "no difference"). If we call the 20- to 29-year-old group population 1, 40- to 49-year-old group population 2, and 60 to 69-year-old group population 3, our null hypothesis would be

$$H_0: \mu_1 = \mu_2 = \mu_3$$

versus the alternative hypothesis

H_1: At least one of the population means is different from the others

If we used an $\alpha = 0.05$ level of significance, each test would have a 95% probability of making a correct decision

$$0.95^3 = 0.86$$

14% is much higher than the desired 5% probability.

To address this problem, Sir Ronald A. Fisher (1890–1962) introduced the method of analysis of variance. The term *analysis of variance* may seem odd since we are conducting a test on means, not variances. However, the name refers to the *approach* we are using, which will involve a comparison of two estimates of the same population variance. The justification for the name will become clear as we develop the test statistic.

Requirements of a One-Way ANOVA Test

1. There are k simple random samples; one from each of k populations.

2. The k samples are independent of each other; that is, the subjects in one group cannot be related in any way to subjects in a second group.

3. The populations are normally distributed.

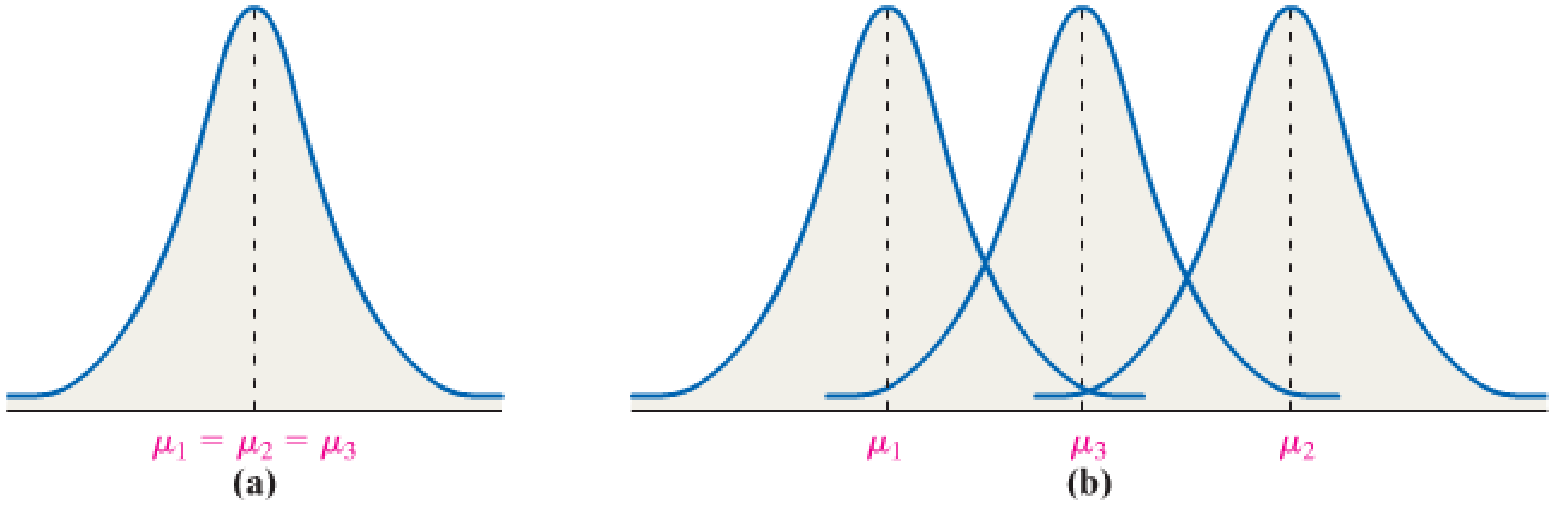
4. The populations have the same variance; that is, each treatment group has population variance σ^2.

The methods of one-way ANOVA are **robust**, so small departures from the requirement of normality will not significantly affect the results of the procedure.

Verifying the Requirement of Equal Population Variances

The one-way ANOVA procedures may be used provided that the largest sample standard deviation is no more than twice the smallest sample standard deviation.

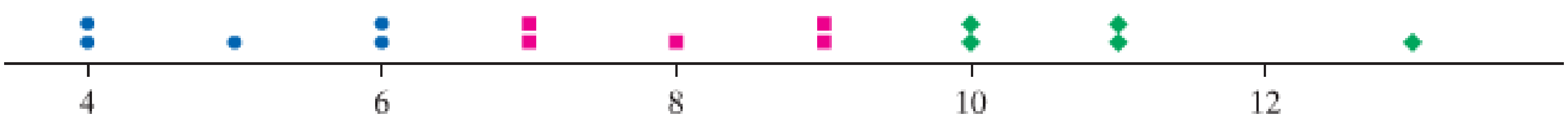
Figure 1(a) shows the distribution of each population if the null hypothesis is true, and Figure 1(b) shows one example of what the distributions of the populations might look like if the alternative hypothesis is true.



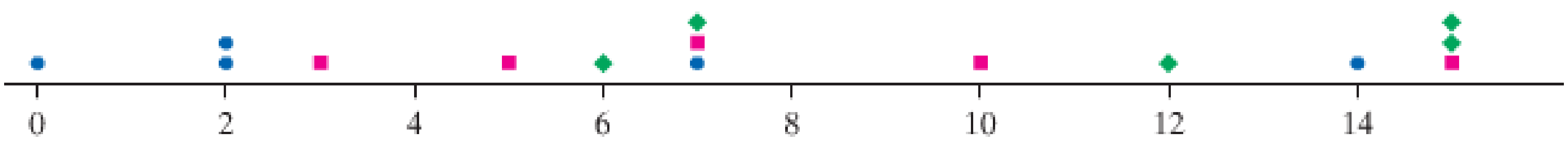
A Conceptual Understanding of One-Way ANOVA

$\overline{x}_1 = \overline{y}_1 = 5, \;\; \overline{x}_2 = \overline{y}_2 = 8, \text{and } \overline{x}_3 = \overline{y}_3 = 11.$

x_1	x_2	x_3	y_1	y_2	y_3
4	7	10	14	10	6
5	8	10	2	3	7
6	9	11	2	15	12
6	7	11	7	7	15
4	9	13	0	5	15
	(a)			**(b)**	



(a)



(b)

ANOVA *F*-Test Statistic

The analysis of variance ***F*-test statistic** is given by

$$F_0 = \frac{\text{between-sample variability}}{\text{within-sample variability}}$$

How to Compute the *F*-Test Statistic

***F*-Test Statistic**

$$F_0 = \frac{\text{mean square due to treatments}}{\text{mean square due to error}} = \frac{\text{MST}}{\text{MSE}}$$

$$\text{MST} = \frac{\text{SST}}{\text{degrees of freedom}} = \frac{n_1(\overline{x}_1 - \overline{x})^2 + n_2(\overline{x}_2 - \overline{x})^2 + \cdots + nk(\overline{x}_k - \overline{x})^2}{k - 1}$$

$$\text{MSE} = \frac{\text{SSE}}{\text{degrees of freedom}} = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \cdots + (n_k - 1)s_k^2}{n - k}$$

x_1	x_2	x_3
4	7	10
5	8	10
6	9	11
6	7	11
4	9	13

(a)

Step 1: We compute the mean of the entire set of data, or the overall mean.

$$\overline{x} = \frac{4 + 5 + 6 + \cdots + 11 + 13}{15} = 8$$

Step 2: Find the sample mean for each treatment (or sample).

$$\overline{x}_1 = 5 \qquad \overline{x}_2 = 8 \qquad \overline{x}_3 = 11$$

Step 3: Find the sample variance for each treatment (or sample).

$$s_1^2 = \frac{(4-5)^2 + (5-5)^2 + (6-5)^2 + (6-5)^2 + (4-5)^2}{5-1} = 1$$

$$s_2^2 = 1 \qquad s_3^2 = 1.5$$

Step 4: Compute the sum of squares due to treatment, SST, and the sum of squares due to error, SSE.

$$\text{SST} = 5(5-8)^2 + 5(8-8)^2 + 5(11-8)^2 = 90$$

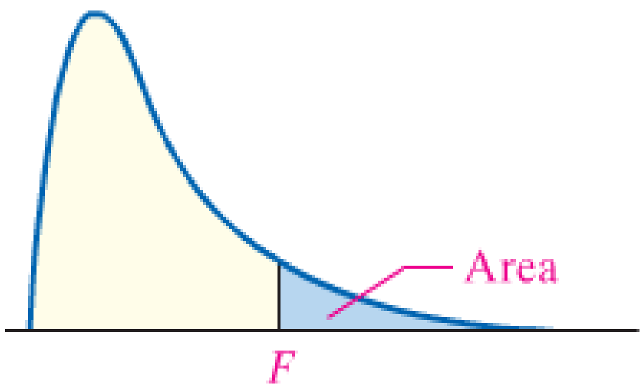
$$\text{SSE} = (5-1)1 + (5-1)1 + (5-1)1.5 = 14$$

Step 5: Compute the means square due to treatment, MST, and the mean square due to error, MSE.

$$\text{MST} = \frac{\text{SST}}{k-1} = \frac{90}{3-1} = 45 \qquad \text{MSE} = \frac{\text{SSE}}{n-k} = \frac{14}{15-3} = 1.1667$$

Step 6: Compute the *F*-test statistic.

$$F_0 = \frac{\text{MST}}{\text{MSE}} = \frac{45}{1.1667} = 38.57$$




```
Fcdf(38.57,1E99,
2,12)
  5.951846167E-6
```

L1	L2	L3 3
4	7	10
5	8	10
6	9	11
6	7	11
4	9	13
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L3(6) =

ANOVA(L1,L2,L3)

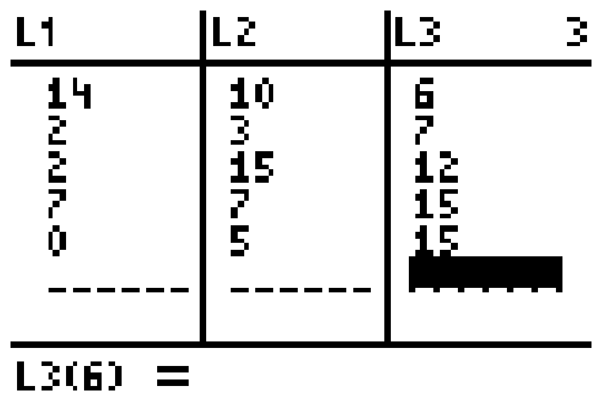
One-way ANOVA

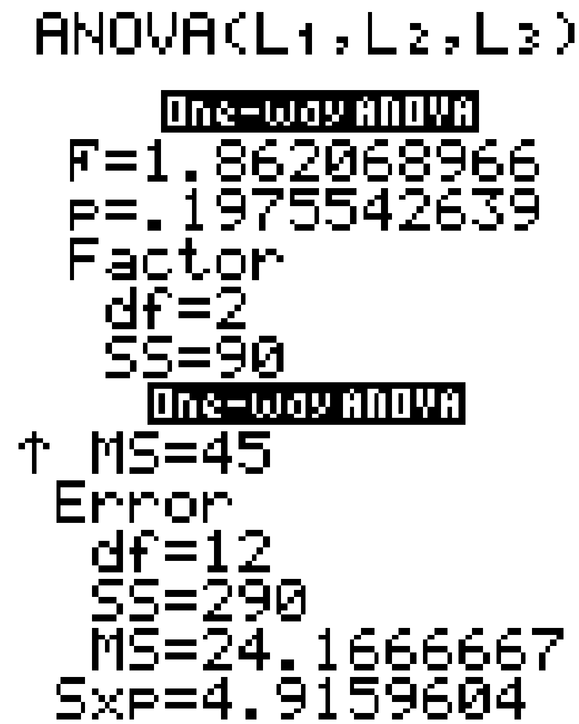
F=38.57142857
p=5.9507017E-6
Factor
df=2
SS=90

One-way ANOVA

↑ MS=45
Error
df=12
SS=14
MS=1.16666667
Sxp=1.08012345

x_1	x_2	x_3	y_1	y_2	y_3
4	7	10	14	10	6
5	8	10	2	3	7
6	9	11	2	15	12
6	7	11	7	7	15
4	9	13	0	5	15
	(a)			(b)	





Problem: The field of prosthodontics is one of nine specialties recognized by the American Dental Association. Prosthodentists specialize in the restoration of oral function, including the use of dental implants, veneers, dentures, and crowns. Since repairing chipped veneer is less time consuming and less costly than complete restoration, a researcher wanted to determine the effect of different repair kits on shear bond strength for repairs of chipped porcelain veneer in fixed prosthodontics. He randomly divided 20 porcelain specimens into four treatment groups. Group 1 specimens used the Cojet system, group 2 used the Silistor system, group 3 used the Cimara system, and group 4 specimens used the Ceramic Repair system. At the conclusion of the study, shear bond strength (in megapascals, MPa) was measured according to ISO 10477. The results presented in Table 1 are based on the results of the study. Verify that the requirements to perform one-way ANOVA are satisfied.

Table 1

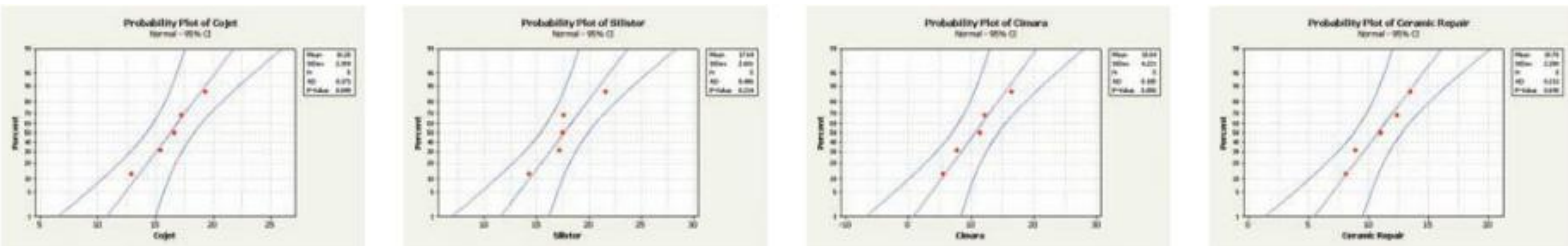
Cojet	Silistor	Cimara	Ceramic Repair
15.4	17.2	5.5	11.0
12.9	14.3	7.7	12.4
17.2	17.6	12.2	13.5
16.6	21.6	11.4	8.9
19.3	17.5	16.4	8.1

Solution

1. The specimens were randomly assigned to the repair systems.

2. None of the specimens is related in any way, so the samples are independent.

3. Figure 2 shows the normal probability plots for all four treatment groups. All of the normal probability plots are roughly linear, so our normality requirement is satisfied.

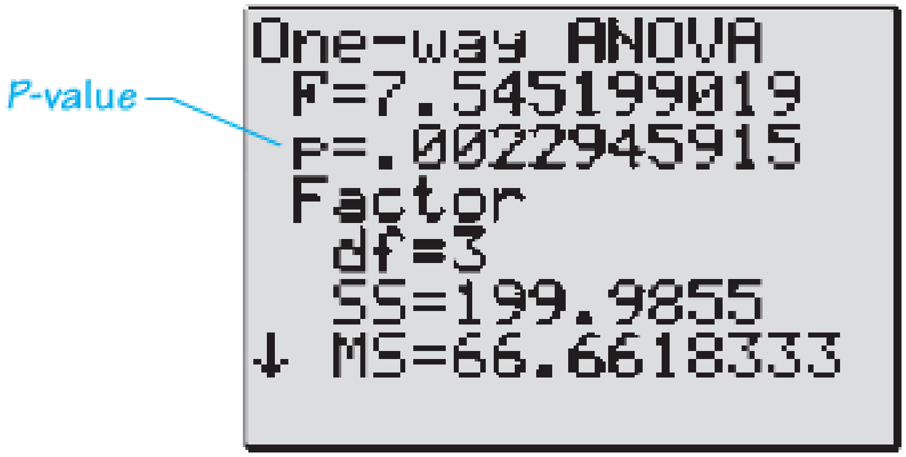


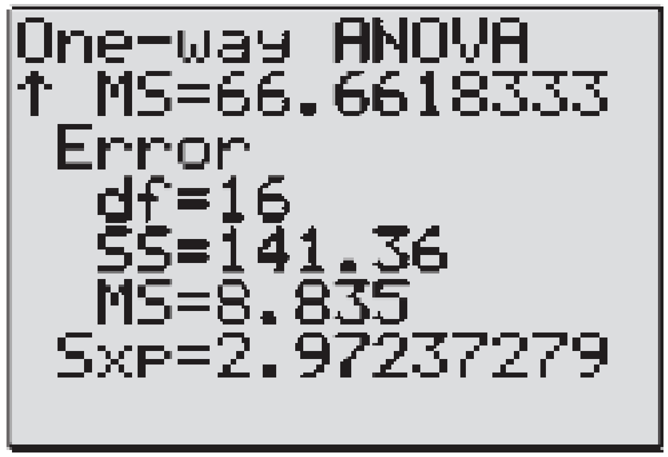
4. The sample standard deviations for each sample are computed using MINITAB and presented as part of Figure 3. The largest standard deviation is 4.22 MPa, and the smallest standard deviation is 2.28 MPa. Because the largest standard deviation is not more than twice the smallest standard deviation ($2 \cdot 2.28 = 4.56 > 4.22$), the requirement of equal population variances is considered satisfied.

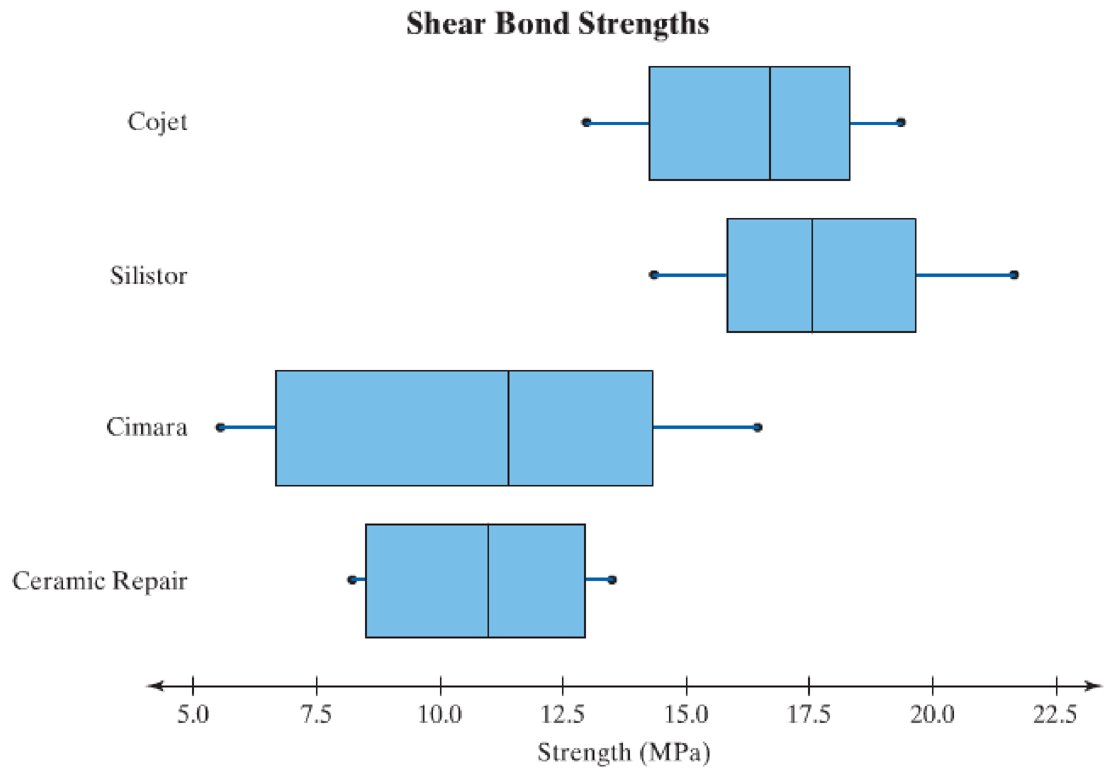
Figure 3

Descriptive Statistics: Cojet, Silistor, Cimara, Ceramic Repair

Variable	Mean	StDev	Variance	Minimum	Q1	Median	Q3	Maximum
Cojet	16.28	2.36	5.57	12.90	14.15	16.60	18.25	19.30
Silistor	17.64	2.60	6.76	14.30	15.75	17.50	19.60	21.60
Cimara	10.64	4.22	17.81	5.50	6.60	11.40	14.30	16.40
Ceramic Repair	10.78	2.28	5.20	8.10	8.50	11.00	12.95	13.50


P-value
One-way ANOVA
F=7.545199019
p=.0022945915
Factor
df=3
SS=199.9855
↓ MS=66.6618333


One-way ANOVA
↑ MS=66.6618333
Error
df=16
SS=141.36
MS=8.835
Sxp=2.97237279


Shear Bond Strengths
Cojet
Silistor
Cimara
Ceramic Repair
5.0
7.5
10.0
12.5
15.0
17.5
20.0
22.5
Strength (MPa)