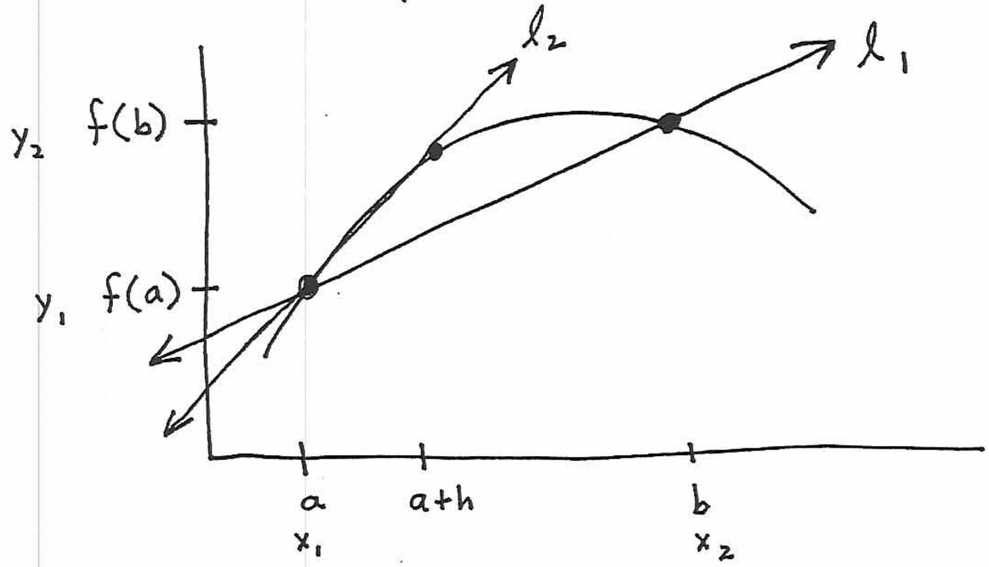
1.4 Derivatives

Definition: A <u>secant line</u> to a curve is a line that intersects the curve at two or more points.

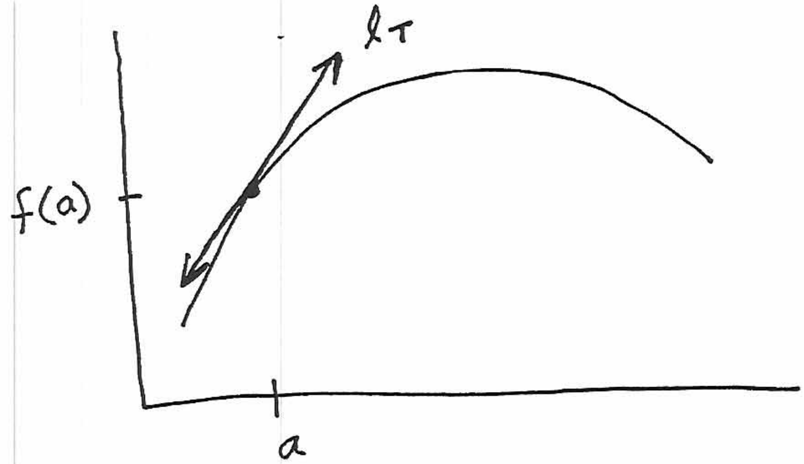


$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$

$$m_2 = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

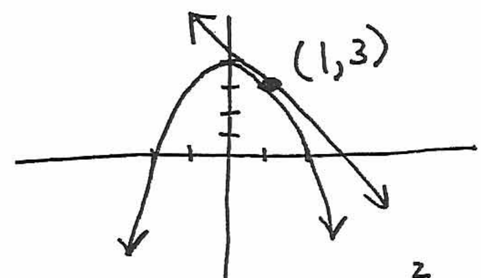
Definition: A <u>tangent line</u> to a curve is a line that best approximates the curve at a single point.

But two points determine a line!!!



$$m_T = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example: Find the equation of the tangent line to the function $f(x) = 4 - x^2$ at the point $(1, 3)$.



$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4 - (1+h)^2 - f(1)}{h}$$

$$= \lim_{h \to 0} \left[\frac{4 - (1 + 2h + h^2) - 3}{h} = \frac{4 - 1 - 2h - h^2 - 3}{h}\right.$$

$$= \frac{3 - 2h - h^2 - 3}{h} = \frac{-2h - h^2}{h} = \frac{\cancel{h}(-2-h)}{\cancel{h}} = -2 - h$$

$$= -2 \quad ; \qquad y - y_1 = m(x - x_1) \Rightarrow y - 3 = -2(x - 1)$$

$$\Rightarrow y = -2x + 5$$

Definition: The generalization of the formula for finding the slope of the tangent line at any given x-coordinate is called the <u>derivative</u>. The derivative of $f(x)$ is often denoted $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Return to the last example. Find the slope of the tangent line to $f(x) = 4 - x^2$ when $x =$ $-2, -1, 0, 1, 2, 5$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{4 - (x+h)^2 - (4 - x^2)}{h}$$

$$= \lim_{h \to 0} \frac{4 - (x^2 + 2xh + h^2) - 4 + x^2}{h}$$

$$= \lim_{h \to 0} \left[\frac{4 - x^2 - 2xh - h^2 - 4 + x^2}{h} = \frac{-2xh - h^2}{h} \right.$$

$$\left. = \frac{\cancel{h}(-2x - h)}{\cancel{h}} \right] = -2x \Rightarrow f'(x) = -2x$$

slopes are $f'(-2)$, $f'(-1)$, $f'(0)$, $\boxed{f'(1)}$, $f'(2)$, $f'(5)$

4 $\quad 2$ $\quad 0$ $\quad \boxed{-2}$ $\quad -4$ $\quad -10$

Ex. Let $f(x) = \frac{2}{x}$, find $f'(x)$

$$\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} = \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \right.$$

$$= \frac{2x - 2(x+h)}{x(x+h)} \cdot \frac{1}{h}$$

$$= \frac{2x - 2x - 2h}{x(x+h)} \cdot \frac{1}{h}$$

$$\left. = \frac{-2\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = \frac{-2}{x(x+h)} \right]$$

$$= \frac{-2}{x \cdot x} = \frac{-2}{x^2}$$

Ex. Let $f(x) = x^3$, find $f'(x)$

$$\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} \right.$$

$$= \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$\left. = 3x^2 + 3xh + h^2 \right] = 3x^2.$$

Ex. Let $f(x) = \sqrt{x}$, find $f'(x)$

$$\lim_{x \to h} \left[\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right.$$

$$= \frac{(x+h) - x}{h} \cdot \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \left. \frac{\cancel{h}}{\cancel{h}} \cdot \frac{1}{\sqrt{x+h} + \sqrt{x}} \right] = \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Differentiability

$f(x)$ does not have a derivative at $x = c$ if at least one of the following is true

① c is a point of discontinuity

② the sign of the slope of the tangent line as $x \to c^-$ is different from the slope of the tangent line as $x \to c^+$

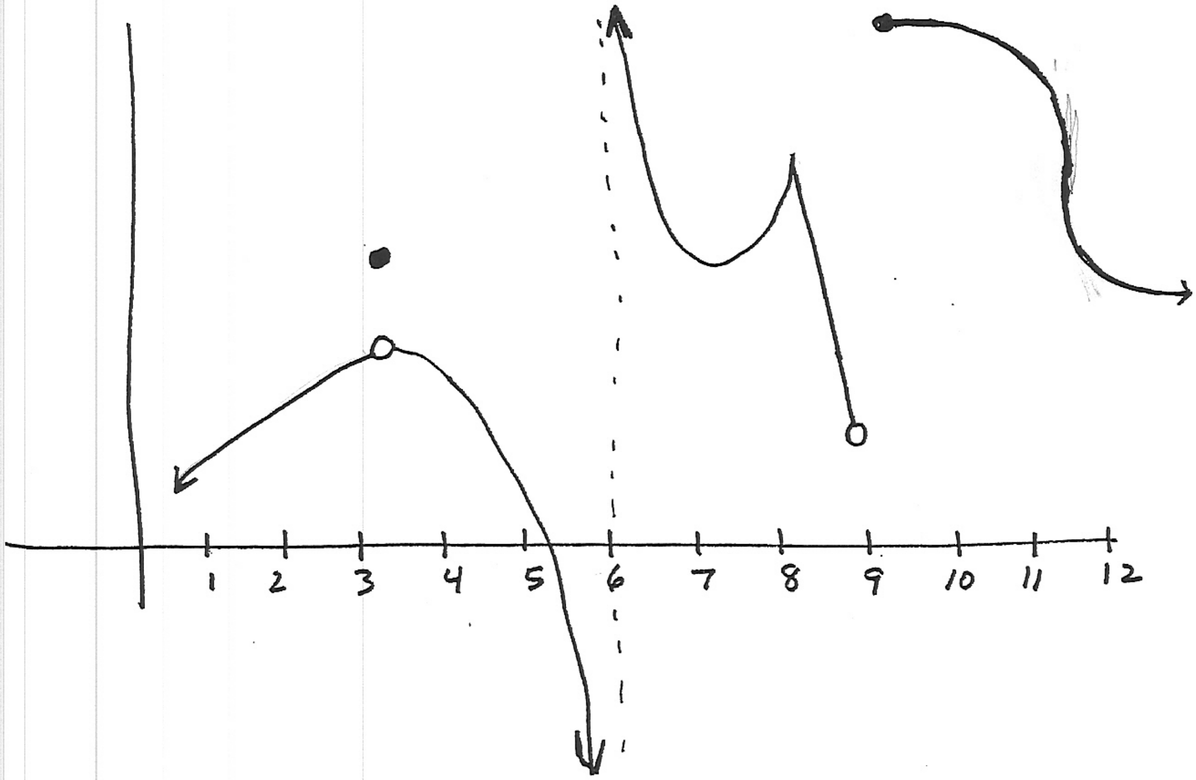
Ex. $f(x) = |x|$ is nondifferentiable at $x = 0$.

(NONDIFFERENTIABLE AT SHARP POINTS)

③ the tangent line to the curve is a vertical line.

Ex. $f(x) = \sqrt[3]{x}$ is nondifferentiable at $x = 0$.

Where is the following function nondifferentiable



note: If $f(x)$ is differentiable at c
then $f(x)$ is continuous at c.