2.1 Increasing and Decreasing Functions

Consider $f(x) = x^3 - 12x + 7$

Sketch it using

xmin = -5	ymin = -25
xmax = 5	ymax = 25
xscl = 1	yscl = 5

Use the $\boxed{\text{draw}}$ feature of your calculator to find the equation of the tangent line when

$x = -3,\ x = -1,\ x = 2,\ x = 3$

$x = -3$: $y = 15x + 61$; $m > 0$ $\Rightarrow$ $f(x)$ increasing on interval containing -3

$x = -1$: $y = -9x + 9$; $m < 0$ $\Rightarrow$ $f(x)$ decreasing on interval containing -1

$x = 2$: $y = -9$; $m = 0$ $\Rightarrow$ 2 is not in an interval where $f(x)$ is increasing or decreasing. Why is $x=2$ an important point in an application?

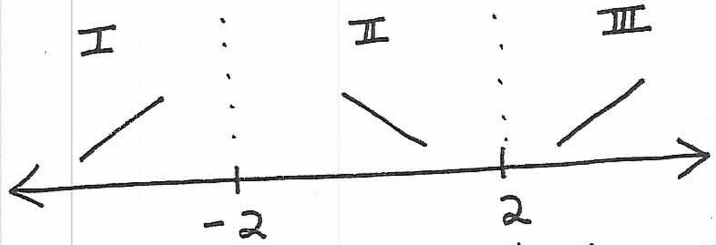
$x = 3$: $y = 15x - 47$; $m > 0$.

Recall, $f'(x)$ is the slope of $f(x)$ at any x if $f'(x)$ exists. For the previous function, find values of x so that $f'(x) = 0$. Then find the open intervals for x where $f(x)$ is increasing and decreasing.

If $f(x) = x^3 - 12x + 7$,

$f'(x) = 3x^2 - 12$; let $f'(x) = 0$ and solve for x.

$3x^2 - 12 = 0 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.



$f'(-3) = 15$ pos. slope

$f'(0) = -12$ neg. slope

$f'(3) = 15$ pos. slope

pick any number from regions

I: $(-\infty, -2)$

II: $(-2, 2)$

III: $(2, \infty)$

SAY $-3, 0, 3$ ← these are called test number

increasing: $x \in (-\infty, -2) \cup (2, \infty)$

decreasing: $x \in (-2, 2)$

Definition: x is a <u>critical number</u> if $x \in D_f$ and $f'(x) = 0$ or $f'(x)$ does not exist.

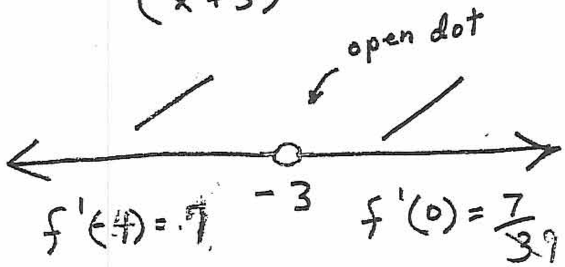
In the previous problem, -2 & 2 were critical numbers. We divided up a number line determined by the critical numbers.

Let $f(x) = \dfrac{x-4}{x+3}$ Find the open intervals for x where $f(x)$ is increasing and decreasing.

<u>Solution</u>

note: $-3 \notin D_f$ so -3 cannot be a critical number but we will use it to divide up our number line.

$$f'(x) = \frac{(x+3)(1) - (x-4)(1)}{(x+3)^2} = \frac{x+3-x+4}{(x+3)^2} = \frac{7}{(x+3)^2}$$

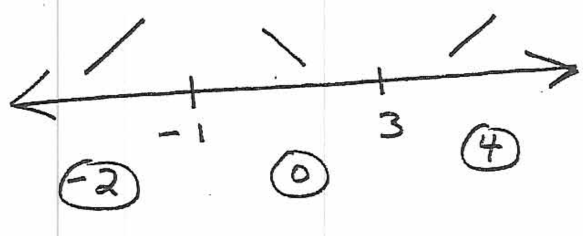


increasing on $(-\infty, -3) \cup (-3, \infty)$; never decreasing

Apply the test to

$f(x) = x^3 + 3x^2 - 9x + 4$

$f'(x) = 3x^2 + 6x - 9 = 3(x^2 - 2x - 3)$
$= 3(x-3)(x+1)$



increasing on $x \in (-\infty, -1) \cup (3, \infty)$
decreasing on $x \in (-1, 3)$

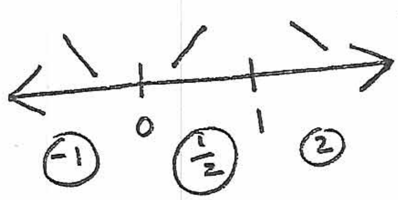
$f(x) = 6x^{2/3} - 4x$

$f'(x) = 4x^{-\frac{1}{3}} - 4$; if $4x^{-\frac{1}{3}} - 4 = 0$
then $x^{-\frac{1}{3}} = 1$
$\left(x^{-\frac{1}{3}}\right)^{-3} = 1^{-3} \Rightarrow x = 1$

$f'(x)$ does not exist if $x = 0$;

$f'(x) = \dfrac{4}{x^{1/3}} - 4.$

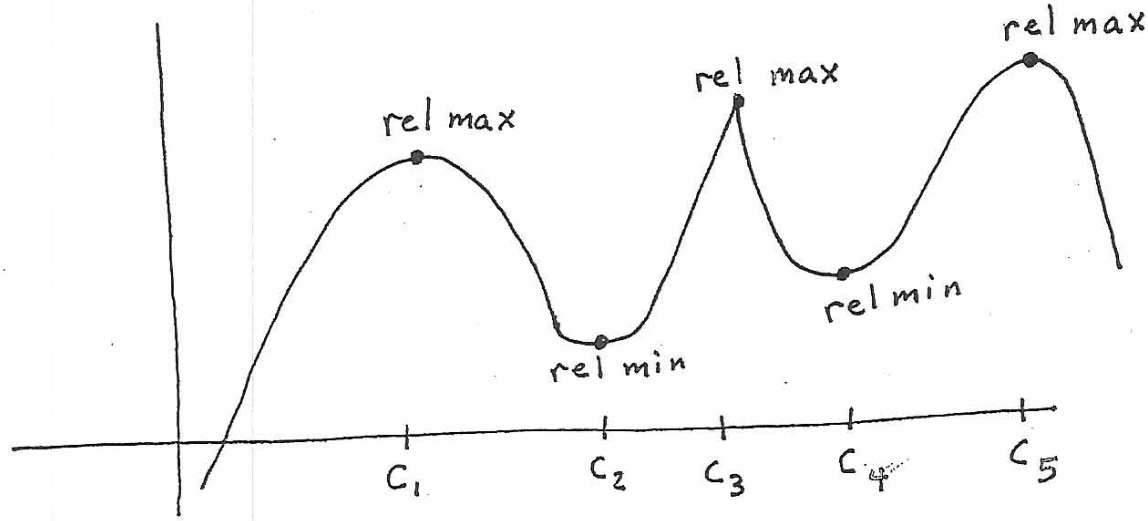
CN: 0, 1



increasing on $(0, 1)$
decreasing on $(-\infty, 0) \cup (1, \infty)$

Relative Extrema

Definition: A function $f(x)$ has a <u>relative extremum</u> at c if $f(c)$ is the largest or smallest value $f(x)$ takes on in an open interval containing c.

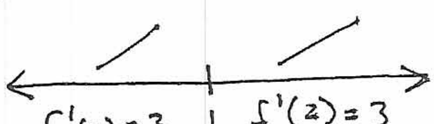


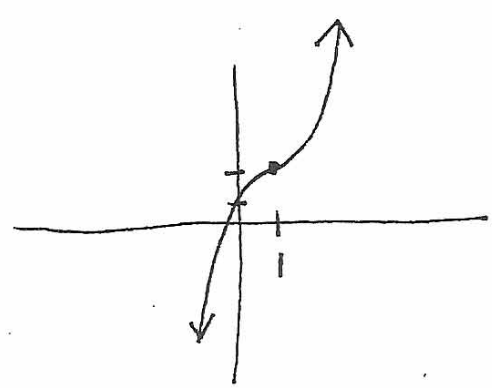
In a nutshell, we are looking for hills, peaks, and vallies.

Observation: If a relative extremum occurs at c, then c must be a critical number. However, if c is a critical number, a relative extremum may not occur at c.

Ex. Let $f(x) = (x-1)^3 + 2$

$f'(x) = 3(x-1)^2$





Conclusion from Observation and Example: $f(x)$ has a relative extremum at c if c is a critical number and c is the only critical number in an open interval where $f'(x)$ changes signs.

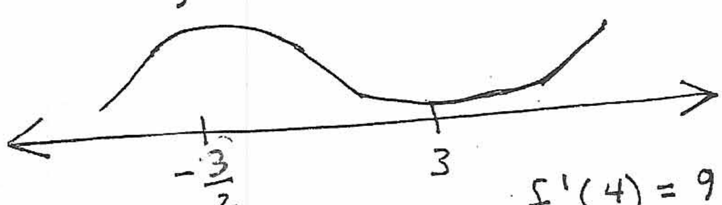
See First Derivative Test on p. 203

Ex. Find Relative Extrema for

$$f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 - 9x - 7$$

$$f'(x) = 2x^2 - 3x - 9 = (2x+3)(x-3)$$

CN: $-\frac{3}{2}$, 3



$f'(-1) = 4$ $\quad$ $f'(0) = -3$ $\quad$ $f'(4) = 9$

Rel Max at $x = -\frac{3}{2} \Rightarrow$ Rel Max is $f(-\frac{3}{2}) = 0.875$

Rel Min at $x = 3 \Rightarrow$ Rel Min is $f(3) = -29.5$

True or False: ~~Is~~ it possible for a relative min to be larger than a relative max?

Ex. Let $f(x) = \dfrac{6\sqrt[3]{(x+4)^2}}{x} + 3$

Note: 0 cannot be a CN since $0 \notin D_f$

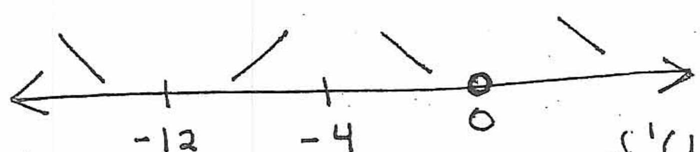
$$f(x) = \frac{6(x+4)^{2/3}}{x} + 3$$

$$f'(x) = \frac{x\left[4(x+4)^{-\frac{1}{3}}\right] - 6(x+4)^{2/3}}{x^2}$$

$$= \frac{2(x+4)^{-\frac{1}{3}}\left[2x - 3(x+4)\right]}{x^2}$$

$$= \frac{2(2x-3x-12)}{(x+4)^{1/3}\,x^2} = \frac{-2(x+12)}{x^2\sqrt[3]{x+4}}$$

CN: $-4, -12$



$f'(-13) < 0$ $\quad f'(-6) > 0 \quad f'(-1) < 0 \quad f'(1) < 0$

Relative min at $x = -12$; $f(-12) = 1$

Relative max at $x = -4$; $f(-4) = 3$

Ex. Let $f(x) = xe^x$. Find the relative extrema.

$$f'(x) = xe^x + e^x = e^x(x+1)$$

CN: -1

$f'(-2) = \frac{-1}{e^2}$ $\quad -1 \quad$ $f'(0) = 2$

Relative Min @ $\left(-1, \frac{-1}{e}\right)$

A hot-air balloon is rising at a rate of 16 ft/sec. A brick is tethered to the side of the balloon. When the balloon is 150 feet above the ground the brick is severed. The height of the brick t seconds after its release is $s(t) = -16t^2 + 16t + 150$.

a) What is the velocity of the brick when $t = .25$ sec.?

b) What is the velocity of the brick when $t = 2$ sec.?

c) What is the max height of the brick?