2.7 Implicit Differentiation

This technique is best when dealing with equations that are not functions; here, y is ~~un~~ usually difficult to solve for.

Ex. Find the slope of the tangent line to the curve $x^2+y^2=25$ at the point $(3,4)$.

not a function

NOTE $\left\{ \begin{array}{l} x^2+y^2=25 \text{ is a circle with center} \\ (0,0) \text{ and radius } 5. \end{array} \right.$

Solve for y: $y^2 = 25 - x^2$

$$y = \pm\sqrt{25-x^2}$$

$$= \pm(25-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \pm\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{\pm x}{\sqrt{25-x^2}} \qquad \text{let } x=3$$

$$m = \frac{\pm 3}{\sqrt{25-9}} = \frac{\pm 3}{\sqrt{16}} = \frac{\pm 3}{4} \quad \left\} \begin{array}{l} \text{Is it } \frac{3}{4} \\ \text{or} \\ -\frac{3}{4}\,? \end{array} \right.$$

$\frac{dy}{dx}$ is the derivative of the function y with respect to the variable x.

Implicit means hidden

y is hidden

$y = \pm\sqrt{25 - x^2}$

$$x^2 + y^2 = 25$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

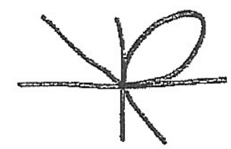
$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

slope at $(3,4)$ is

$$m = \frac{-x}{y} = \frac{-3}{4}.$$

Find the equation of the tangent line at (2,4) for the curve
$x^3 + y^3 = 9xy$ [Folium of Descartes]



$$3x^2 + 3y^2 \frac{dy}{dx} = 9x \frac{dy}{dx} + 9y$$

$$3y^2 \frac{dy}{dx} - 9x \frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} (3y^2 - 9x) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3(3y - x^2)}{3(y^2 - 3x)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

$$m = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{12 - 4}{16 - 6} = \frac{8}{10} = \frac{4}{5}$$

$$y - 4 = \frac{4}{5}(x - 2) \Rightarrow y - 4 = \frac{4}{5}x - \frac{8}{5}$$

$$+\frac{20}{5} \qquad +\frac{20}{5}$$

$$y = \frac{4}{5}x + \frac{12}{5}$$

Ex. $xy + y^2 = 6$ at $(1, 2)$

Related Rates: $\frac{df}{dt}$

Ex. Let $8y^3 + x^2 = 1$;

$\frac{dx}{dt} = 2$, $x = 3$, $y = -1$.

Find $\frac{dy}{dt}$.

Solution: Here, x and y are functions of time.

$$\left.\begin{array}{l} x(t) = ? \\ y(t) = ? \end{array}\right\} \text{what they are is hidden}$$

We will use implicit differentiation.

$8y^3 + x^2 = 1 \Rightarrow$

$24y^2 \cdot \frac{dy}{dt} + 2x\frac{dx}{dt} = 0$; plug in values

$24(-1)^2 \cdot \frac{dy}{dt} + 2(3)(2) = 0 \Rightarrow$

$24\frac{dy}{dt} + 12 = 0 \Rightarrow \frac{dy}{dt} = -\frac{1}{2}$

Related Rates

Ex. A pebble is dropped into a pond and concentric circles form. If the radius is increasing at 8 inches per seconds, how fast is the area of the outer circle changing when the radius of the outer circle is 60 inches?

Solution

$$A = \pi r^2 \Rightarrow$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$= 2\pi(60\text{in})(8\text{ in/sec})$$

$$= 960\pi \text{ in}^2/\text{sec}$$

Ex. All edges of an ice cubes are melting at a rate of 5 mm per minute. How fast is the volume of the cube changing when the lengths of the edges are 1 cm?

$$V = x^3$$

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$= 3(10\text{ mm})^2(-5\text{ mm/min})$$

$$= -1500 \text{ mm}^3/\text{min}$$

or

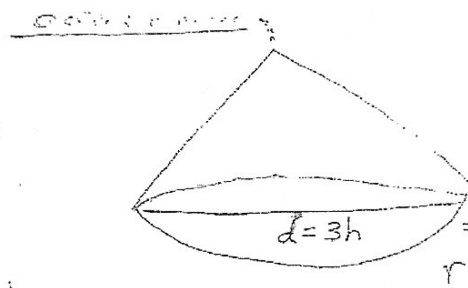
$$-1.5 \text{ cm}^3/\text{min}$$

Ex. A 5-meter ladder resting on a wall begins to slip. The top of the ladder is falling at a rate of 1.5 m/sec when the bottom of the ladder is 3 meters from the wall. At this time, how fast is the bottom of the ladder moving away from the wall?

X. At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 ft^3/min. The diameter of the base is three times its height. At what rate is the height changing when it is 15 feet high?

Solution

$$\frac{dV}{dt} = 10\ ft^3/min$$



$\Rightarrow$ $r = \frac{3h}{2}$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{3h}{2}\right)^2 h$$

$$V = \frac{3}{4}\pi h^3$$

$$\frac{dV}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt} \quad \Longrightarrow$$

$$\frac{dh}{dt} = \frac{4\frac{dV}{dt}}{9\pi h^2} = \frac{4(10\ ft^3/min)}{9\pi(15\ ft)^2}$$

$$= \frac{40}{2025\pi}\ ft/min$$

$$= \frac{8}{405\pi}\ ft/min$$

$$\approx .0063\ ft/min$$