3.1, 3.2, + 3.5 Derivatives of Exponential Functions

let $h = \frac{1}{k}$

$\Longleftrightarrow$

Recall, $e = \left(1+\frac{1}{k}\right)^{k}$, $k \to \infty$

$e = (1+h)^{\frac{1}{h}}$, $h \to 0$

Theorem: If $f(x) = e^{x}$, $f'(x) = e^{x}$

Proof

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} = \frac{e^{x+h}-e^{x}}{h} = \frac{e^{x}e^{h}-e^{x}}{h} = \frac{e^{x}(e^{h}-1)}{h}\right];$$

$$e^{h} = \left[(1+h)^{\frac{1}{h}}\right]^{h} \quad \text{as } h \to 0$$

$$e^{h} = 1+h$$

$$f'(x) = \lim_{h\to 0}\left[\frac{e^{x}\left[(1+h)-1\right]}{h} = \frac{he^{x}}{h} = e^{x}\right] = e^{x} \ \blacksquare$$

Theorem: If $y = e^{g(x)}$ then

$$\frac{dy}{dx} = e^{g(x)} \cdot g'(x)$$

this can be easily proven by letting $u = g(x)$ then using the chain rule $\left[\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \right]$

Ex. If $f(x) = e^{4x}$ then

$f'(x) =$

Ex. If $f(x) = e^{x^2 - x}$ then

$f'(x) =$

If $f(x) = 2^x$, find $f'(x)$.

Recall

$$\boxed{\begin{array}{l} ① \ e^{\ln x} = x \\ ② \ \ln a^n = n \ln a \end{array}}$$

$$f(x) = 2^x = e^{\ln 2^x} = e^{x \ln 2}$$

$\ln 2 \approx .693$
its a constant

$$f'(x) = (\ln 2) e^{x \ln 2} = (\ln 2) 2^x$$

Theorem: If $f(x) = a^x$
then $f'(x) = (\ln a) a^x$

Theorem: If $f(x) = a^{g(x)}$
then $f'(x) = (\ln a) a^{g(x)} g'(x)$

Ex. Let $f(x) = 7 \cdot 3^{4-x^3}$. Find $f'(x)$

$$f'(x) = 7\left[(\ln 3)\, 3^{4-x^3}(3x^2)\right]$$

$$= 21x^2 (\ln 3)\, 3^{4-x^3}$$

Ex. Let $f(x) = (3x-2)^5 e^{7x}$

$$f'(x) = (3x-2)^5\left[7e^{7x}\right] + e^{7x}\left[5(3x-2)^4(3)\right]$$

$$= e^{7x}(3x-2)^4\left[7(3x-2) + 15\right]$$

$$= e^{7x}(3x-2)^4(21x+1)$$

Theorem: $\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\left\{ \frac{d}{dx}(\ln|x|) = \frac{1}{x} \right.$

Theorem: $\frac{d}{dx}\left[\ln|g(x)|\right] = \frac{g'(x)}{g(x)}$

Ex. Let $f(x) = \ln(8x^2 - 7x + 4)$. Find $f'(x)$.

$$f'(x) = \frac{16x - 7}{8x^2 - 7x + 4}$$

Ex. Let $f(x) = x \ln x$. Find $f'(x)$

$$f'(x) = x \cdot \frac{1}{x} + \ln x$$

$$= 1 + \ln x$$

Ex. Let $f(x) = \ln(kx)$ where k is a constant

$$f'(x) = \frac{k}{kx} = \frac{1}{x}$$

Ex. $f(x) = x \log_3 (2x-1)$

$$f'(x) = \frac{2x}{(\ln 3)(2x-1)} + \log_3 (2x-1)$$

Ex. $f(x) = \ln (17x^2-1)^{15}$

$$f(x) = 15 \ln (17x^2-1)$$

$$f'(x) = 15 \cdot \frac{34x}{17x^2-1} = \frac{510x}{17x^2-1}$$

Ex. $f(x) = \ln \frac{2x-3}{5x+6}$

$$f(x) = \ln (2x-3) - \ln (5x+6)$$

$$f'(x) = \frac{2}{2x-3} - \frac{5}{5x+6}$$

Ex. $f(x) = \ln \left[(2x+1)^7 (7x^2-1)^3\right]$

$$f(x) = 7 \ln (2x+1) + 3 \ln (7x^2-1)$$

$$f'(x) = 7 \cdot \frac{2}{2x+1} + 3 \cdot \frac{14x}{7x^2-1}$$

$$= \frac{14}{2x+1} + \frac{42x}{7x^2-1}$$

Recall, $\log_a x = \dfrac{\ln x}{\ln a}$ (Change-of-base Rule)

Ex. Find $\log_7 100$

then check answer.

Ex. If $f(x) = \log_7 x$ find $f'(x)$.

$$f(x) = \frac{\ln x}{\ln 7} \implies f'(x) = \frac{\frac{1}{x}}{\ln 7} = \frac{1}{(\ln 7)\,x}$$

Theorem: $\dfrac{d}{dx}\left(\log_a |x|\right) = \dfrac{1}{(\ln a)\,x}$

Theorem: $\dfrac{d}{dx}\left[\log_a |g(x)|\right] = \dfrac{g'(x)}{(\ln a)\,g(x)}$

Find $\frac{dy}{dx}$, if $y = \log_4 \frac{x^3 + 7x}{4x^2 - 9}$

$$y = \log_4 (x^3 + 7x) - \log_4 (4x^2 - 9)$$

$$\frac{dy}{dx} = \frac{3x^2 + 7}{(\ln 4)(x^3 + 7x)} - \frac{8x}{(\ln 4)(4x^2 - 9)}$$

Find $\frac{dy}{dx}$ if $y = \frac{7x\sqrt{2x+1}}{\sqrt[3]{7x^2 - 5}}$