Fundamental Theorem of Calculus 4.3
Riemann's Proof

If $f(x)$ is continuous and $f(x) \geq 0$ on $[a, b]$ then the area under the curve $f(x)$ from a to b is $F(b) - F(a)$

$$A = \int_a^b f(x)\,dx = F(b) - F(a) \quad \boxed{\text{Definite Integral}}$$

Proof: $\text{Area} = \Delta x \sum_{i=0}^{n-1} f(x_i)$ where

$a = x_0 < x_1 < x_2 < \ldots < x_n = b$ and $\Delta x = \frac{b-a}{n}$; $n \to \infty$

Let $F(x)$ be the antiderivative; $f(x) = F'(x)$.

$$f(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$\text{Area} = \Delta x \sum_{i=0}^{n-1} \lim_{\Delta x \to 0} \frac{F(x_i + \Delta x) - F(x_i)}{\Delta x}$$

$$= \sum_{i=0}^{n-1} \lim_{\Delta x \to 0} \left[F(x_i + \Delta x) - F(x_i)\right]$$

$$= \lim_{\Delta x \to 0} \Big\{ \Big[F(\overbrace{\cancel{x_0} + \Delta x}^{x_1}) - F(\overbrace{x_0}^{a})\Big] + \Big[F(\overbrace{\cancel{x_1} + \Delta x}^{x_2}) - F(\cancel{x_1})\Big] +$$

$$\Big[F(\overbrace{x_2 + \Delta x}^{x_3}) - F(\cancel{x_2})\Big] + \ldots \Big[F(\overbrace{\cancel{x_{n-2}} + \Delta x}^{x_{n-1}}) -$$

$$F(x_{n-2})\Big] + \Big[F(\overbrace{x_{n-1} + \Delta x}^{x_n = b}) - F(\cancel{x_{n-1}})\Big] \Big\} =$$

$$-F(a) + F(b) = F(b) - F(a).$$

$\boxed{\text{EOP}}$

Ex. Find the area between the x-axis and $f(x) = 3x^2 + 2x + 1$;

that is, find

$$A = \int_0^2 (3x^2 + 2x + 1)\, dx = F(2) - F(0)$$

Find

$$\int_0^{\ln 5} e^{4x}\,dx$$

$$\int_e^{e^3} \left(\frac{x^2+1}{x} \right) dx$$

$$\int_0^2 (x^2-4)\,dx$$

Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$ on $[-1, 2]$.

Ex. If the marginal revenue derived by selling x widgets is $R'(x) = 3.2x^{0.14}$ dollars. Find the revenue derived by selling 900 widgets.

At 10:00 AM, the speed of a truck after t minutes can be described by $r(t) = 100t^2 - 90t$ $0 \le t \le 10$ in feet/min.

How many miles did the truck travel between 10:05 & 10:09?