The line $x = a$ is a **vertical asymptote** if any of the following limit statements are true:

$$\lim_{x\to a^-} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty \quad \text{or}$$

$$\lim_{x\to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty.$$

Determine the vertical asymptote(s) of the function.

$$* \quad f(x) = \frac{9x-7}{x-4}$$

$x - 4 = 0 \Rightarrow$ $x = 4$ is the VA

HA: $y = 9$

Determine the vertical asymptote(s) of the function.

$$f(x) = \frac{x-4}{x^3 - 10x^2 + 24x} = \frac{x-4}{x(x^2-10x+24)} = \frac{\cancel{x-4}^{\,1}}{x\cancel{(x-4)}(x-6)}$$

$$= \frac{1}{x(x-6)} \Rightarrow \text{VA: } x = 0 \ (y\text{-axis}), \ x = 6$$

Determine the vertical asymptote(s) of the function.

$$f(x) = \frac{2}{x^2+4} \; ;$$

$x^2 + 4 = 0$

$x^2 = -4 \Rightarrow x = \pm\sqrt{-4}$

$= \pm 2i$

NO VA

The line $y = b$ is a **horizontal asymptote** if either or both of the following limit statements are true:

$$\lim_{x \to -\infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to \infty} f(x) = b.$$

Determine the horizontal asymptote of the function.

$$f(x) = \frac{2x}{12x - 7} \quad ; \quad HA: y = \frac{2}{12} \Rightarrow y = \frac{1}{6}$$

Determine the horizontal asymptote of the function.

$$f(x) = \frac{2x}{x^2 - 7x} = \frac{0x^2 + 2x}{x^2 - 7x} \quad ; \quad HA: y = \frac{0}{1} = 0$$

$y = 0$ (x-axis)

Determine the horizontal asymptote of the function.

$$f(x) = \frac{2x^4 - 3x^2}{8x^3 + 5x^2} = \frac{2x^4 - 3x^2}{0x^4 + 8x^3 + 5x^2} \quad ; \quad HA: y = \frac{2}{0}$$

NO HA

Determine the horizontal asymptote of the function.

$$f(x) = \frac{9x^3 - 2x + 3}{12x^3 + 7x - 8}$$

GRAPH

$$f(x)=\frac{-2}{x-3} = \frac{0x-2}{x-3}$$

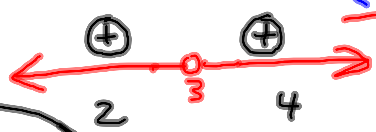
① x- and y-intercept
② Asymptotes
③ inc/dec
④ rel. ext.
⑤ concavity / infl. pts

① x-int: Solve $f(x)=0$; $-2\neq 0 \Rightarrow$ no x-int.
y-int: find $f(0)=\frac{-2}{-3}=\frac{2}{3} \Rightarrow (0, 2/3)$

② VA: $x=3$, HA: $y=0$ (x-axis)

③ $f(x) = -2(x-3)^{-1} \Rightarrow f'(x) = 2(x-3)^{-2}(1)$
$= \frac{2}{(x-3)^2}$

CV: None
pt of disc: 3

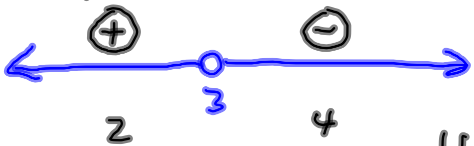


inc: $(-\infty, 3) \cup (3, \infty)$

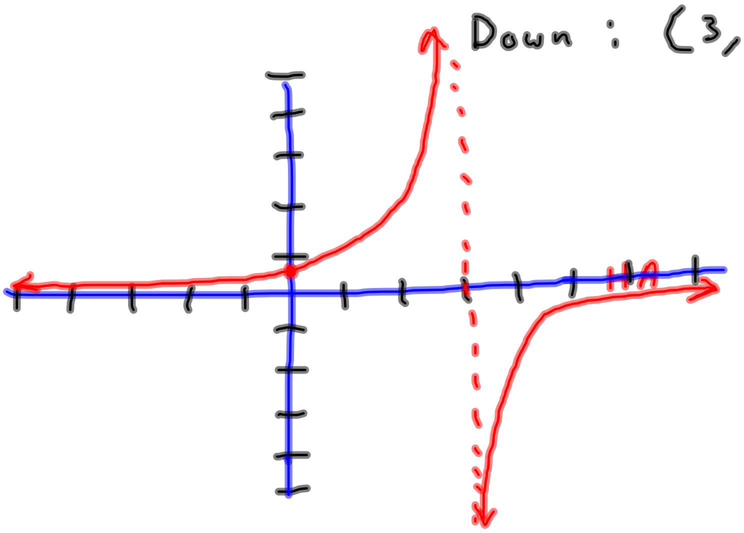
④ No CV's , No Rel. Ext.

⑤ $f'(x) = 2(x-3)^{-2} \Rightarrow f''(x) = -4(x-3)^{-3}(1)$
$= \frac{-4}{(x-3)^3}$

Upward   Downward



Up: $(-\infty, 3)$
Down: $(3, \infty)$



$$f(x) = \frac{-1}{x^2+4}$$

① No x-int

$(0, -\frac{1}{4})$

② NO VA ; HA: $y=0$ (x-axis)

③ $f(x) = -1(x^2+4)^{-1} \Rightarrow f'(x) = (x^2+4)^{-2}(2x)$

$$f'(x) = \frac{2x}{(x^2+4)^2}$$

CV: 0

$\ominus$ $\oplus$

$\boxed{-1}$ 0 $\boxed{1}$

dec: $(-\infty, 0)$

inc: $(0, \infty)$

④ $f'(x) = \dfrac{2x}{(x^2+4)^2} \Rightarrow$

$$f''(x) = \frac{(x^2+4)^2(2) - 2x\left[2(x^2+4)(2x)\right]}{(x^2+4)^4}$$

$$= \frac{2(x^2+4)^2 - 8x^2(x^2+4)}{(x^2+4)^4}$$

$$= \frac{2\cancel{(x^2+4)}\left[(x^2+4) - 4x^2\right]}{(x^2+4)^{\cancel{4}}}$$

$$= \frac{2(4-3x^2)}{(x^2+4)^3}$$

$4 - 3x^2 = 0 \Rightarrow$

$3x^2 = 4 \Rightarrow$

$x^2 = \frac{4}{3} \Rightarrow$

$x = \pm \frac{2}{\sqrt{3}}$

CV'': $\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}$

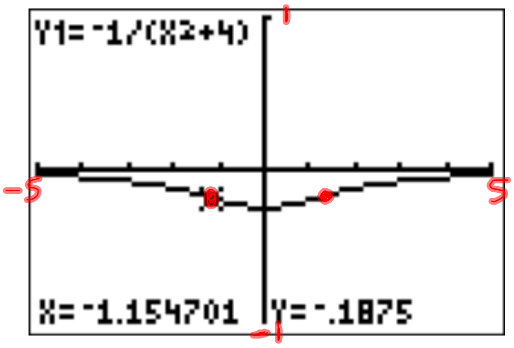
$\ominus$ $\oplus$ $\ominus$

$\boxed{-2}$ $\frac{-2}{\sqrt{3}}$ $\boxed{0}$ $\frac{2}{\sqrt{3}}$ $\boxed{2}$

-1.15 1.15

Concave Downward: $(-\infty, \frac{-2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, \infty)$

Concave Upward: $(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$



Find the limit, if it exists.

$$\lim_{x\to\infty}\left[\frac{-8x^4+6x}{8x^3-4x-8}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{-8x+\frac{6}{x^2}}{8-\frac{4}{x^2}-\frac{8}{x^3}}\right] =$$

$$\frac{-8(\infty)+0}{8-0-0} = (-1)(\infty) = -\infty$$

$$\lim_{x\to-\infty} f(x) = \frac{-8(-\infty)+0}{8-0-0} = (-1)(-\infty) = \infty$$

Find the limit, if it exists.

$$\lim_{x\to-\infty}\left[\frac{2x^2+x}{x+2}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \frac{2x+1}{1+\frac{2}{x}}\right] =$$

$$\frac{2(-\infty)+1}{1+0} = -\infty$$

DEFINITION:

A linear asymptote that is neither vertical nor horizontal is called a **slant**, or **oblique**, **asymptote**. For any rational function of the form $f(x) = p(x)/q(x)$, a slant asymptote occurs when the degree of $p(x)$ is exactly 1 more than the degree of $q(x)$. A graph can cross a slant asymptote.

$$
\begin{array}{r|rrr}
1 & 1 & 0 & -4 \\
 & & 1 & 1 \\
\hline
 & 1 & 1 & \boxed{-3}
\end{array}
$$

$y = x + 1$

Find the slant asymptote:

$$f(x) = \frac{x^2 - 4}{x - 1}$$

$$
\begin{array}{r@{\,}l}
 & x + 1 + \dfrac{-3}{x-1} \\
x - 1 & \overline{)\,x^2 + 0x - 4} \\
 & -x^2 + x \\
\hline
 & \quad x - 4 \\
 & \quad -x + 1 \\
\hline
 & \qquad -3
\end{array}
$$

$y = x + 1$