## Limits: A Numerical and Graphical Approach

1.1

*OBJECTIVE*

- Find limits of functions, if they exist, using numerical or graphical methods.




## 1.1 Limits: A Numerical and Graphical Approach

**DEFINITION:**

As $x$ approaches $a$, the **limit** of $f(x)$ is $L$, written

$$\lim_{x \to a} f(x) = L,$$

if all values of $f(x)$ are close to $L$ for values of $x$ that are sufficiently close, but not equal to, $a$.




## 1.1 Limits: A Numerical and Graphical Approach

**THEOREM:**

As $x$ approaches $a$, the **limit** of $f(x)$ is $L$, if the limit from the left exists and the limit from the right exists and both limits are $L$. That is, if

1) $\lim_{x \to a^-} f(x) = L,$

and

2) $\lim_{x \to a^+} f(x) = L,$

then

$$\lim_{x \to a} f(x) = L,$$




## 1.1 Limits: A Numerical and Graphical Approach

Quick Check 1

Let $f(x) = \dfrac{x^2 - 9}{x - 3}$

a) What is $f(3)$?

b) What is the limit of $f$ as $x$ approaches 3?

## 1.1 Limits: A Numerical and Graphical Approach

Quick Check 1 Solution a)

1.) Since $f(x)=\dfrac{x^2-9}{x-3}$, we will substitute 3 in for $x$, giving us the new equation $f(3)=\dfrac{3^2-9}{3-3}$.

2.) Solving for $f(3)$, we get $f(3)=\dfrac{3^2-9}{3-3}=\dfrac{9-9}{3-3}=\dfrac{0}{0}$.

Thus $f(3)$ does not exist.




## 1.1 Limits: A Numerical and Graphical Approach

Quick Check 1 Solution b)

First let $x$ approach 3 from the left:

| $x\to 3^-$ | 2 | 2.5 | 2.9 | 2.99 | 2.999 |
|---|---|---|---|---|---|
| $f(x)$ | 5 | 5.5 | 5.9 | 5.99 | 5.999 |

Thus it appears that $\lim_{x\to 3^-} f(x)$ is 6.

Next let $x$ approach 3 from the right:

| $x\to 3^+$ | 4 | 3.5 | 3.1 | 3.01 | 3.001 |
|---|---|---|---|---|---|
| $f(x)$ | 7 | 6.5 | 6.1 | 6.01 | 6.001 |

Thus it appears that $\lim_{x\to 3^+} f(x)$ is 6.

Since both the left-hand and right-hand limits agree, $\lim_{x\to 3} f(x)=6$.




## 1.1 Limits: A Numerical and Graphical Approach

**Example 1:** Consider the function $H$ given by

$$H(x)=\begin{cases}2x+2 & \text{for } x<1\\ 2x-4 & \text{for } x\geq 1\end{cases}$$

Graph the function, and find each of the following limits, if they exist. When necessary, state that the limit does not exist.

a) $\lim_{x\to 1} H(x)$ b) $\lim_{x\to -3} H(x)$




## 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically**

First, let $x$ approach 1 from the left:

| $x\to 1^-$ | 0 | 0.5 | 0.8 | 0.9 | 0.99 | 0.999 |
|---|---|---|---|---|---|---|
| $H(x)$ | 2 | 3 | 3.6 | 3.8 | 3.98 | 3.998 |

Thus, it appears that $\lim_{x\to 1^-} H(x)=4$.

## 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically (continued)**

Then, let $x$ approach 1 from the right:

| $x \to 1^+$ | 2 | 1.8 | 1.1 | 1.01 | 1.001 | 1.0001 |
|---|---|---|---|---|---|---|
| $H(x)$ | 0 | –0.4 | –1.8 | –1.98 | –1.998 | –1.9998 |

Thus, it appears that $\lim_{x \to 1^+} H(x) = -2$.




## 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically (concluded)**

Since 1) $\lim_{x \to 1^-} H(x) = 4$

and

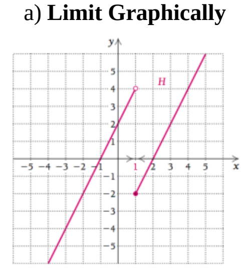
2) $\lim_{x \to 1^+} H(x) = -2$

Then, $\lim_{x \to 1} H(x)$ does not exist.




## 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Graphically**




Observe on the graph that:

1) $\lim_{x \to 1^-} H(x) = 4$

and

2) $\lim_{x \to 1^+} H(x) = -2$

Therefore,

$\lim_{x \to 1} H(x)$ does not exist.




## 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically**

First, let $x$ approach –3 from the left:

| $x \to -3^-$ | –4 | –3.5 | –3.1 | –3.01 | –3.001 |
|---|---|---|---|---|---|
| $H(x)$ | –6 | –5 | –4.2 | –4.02 | –4.002 |

Thus, it appears that $\lim_{x \to -3^-} H(x) = -4$.

## 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically (continued)**

Then, let $x$ approach –3 from the right:

| $x \to -3^+$ | –2 | –2.5 | –2.9 | –2.99 | –2.999 |
|---|---|---|---|---|---|
| $H(x)$ | –2 | –3 | –3.8 | –3.98 | –3.998 |

Thus, it appears that $\lim_{x \to -3^+} H(x) = -4$.




## 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically (concluded)**

Since 1) $\lim_{x \to -3^-} H(x) = -4$

and

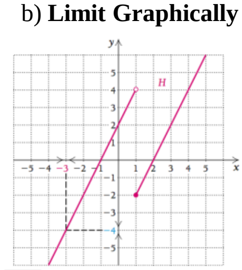
2) $\lim_{x \to -3^+} H(x) = -4$

Then, $\lim_{x \to -3} H(x) = -4$.




## 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Graphically**




Observe on the graph that:

1) $\lim_{x \to -3^-} H(x) = -4$

and

2) $\lim_{x \to -3^+} H(x) = -4$

Therefore,

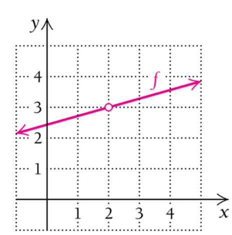
$$\lim_{x \to -3} H(x) = -4.$$




## 1.1 Limits: A Numerical and Graphical Approach

Quick Check 2

Calculate the following limits based on the graph of $f$.




a.) $\lim_{x \to 2^-} f(x)$

b.) $\lim_{x \to 2^+} f(x)$

c.) $\lim_{x \to 2} f(x)$

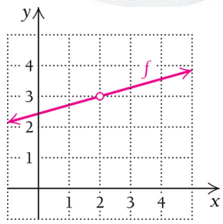
## 1.1 Limits: A Numerical and Graphical Approach

Quick Check 2 Solution

a.) $\lim_{x\to 2^-} f(x)$ : By looking at the graph, as $x$ approaches $2$ from the left, we can see that the $\lim_{x\to 2^-} f(x) = 3$.

b.) $\lim_{x\to 2^+} f(x)$ : By looking at the graph, as $x$ approaches $2$ from the right, we can see that the $\lim_{x\to 2^+} f(x) = 3$.

c.) Based on the solutions to parts a.) and b.), we know that the $\lim_{x\to 2} f(x) = 3$.







## 1.1 Limits: A Numerical and Graphical Approach

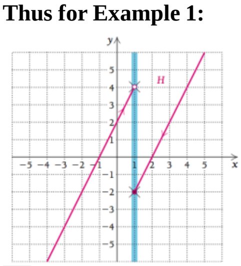
**The "Wall" Method:**

As an alternative approach to Example 1, we can draw a "wall" at $x = 1$, as shown in blue on the following graphs. We then follow the curve from left to right with pencil until we hit the wall and mark the location with an ×, assuming it can be determined. Then we follow the curve from right to left until we hit the wall and mark that location with an ×. If the locations are the same, we have a limit. Otherwise, the limit does not exist.

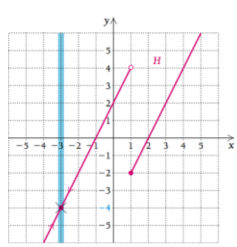



## 1.1 Limits: A Numerical and Graphical Approach

**Thus for Example 1:**




$\lim_{x\to 1} H(x)$ does not exist




$\lim_{x\to -3} H(x) = -4$




## 1.1 Limits: A Numerical and Graphical Approach

**Example 2:** Consider the function $f$ given by

$$f(x) = \frac{1}{x-2} + 3$$

Graph the function, and find each of the following limits, if they exist. If necessary, state that the limit does not exist.

a) $\lim_{x\to 3} f(x)$ b) $\lim_{x\to 2} f(x)$

## 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically**

Let $x$ approach 3 from the left and right:

| $x \to 3^-$ | 2.1 | 2.5 | 2.9 | 2.99 |
|---|---|---|---|---|
| $f(x)$ | 13 | 5 | $4.\overline{11}$ | $4.\overline{01}$ |

$\Rightarrow \lim_{x \to 3^-} f(x) = 4$

| $x \to 3^+$ | 3.5 | 3.2 | 3.1 | 3.01 |
|---|---|---|---|---|
| $f(x)$ | $3.6\overline{6}$ | $3.8\overline{3}$ | $3.90\overline{90}$ | $3.\overline{9900}$ |

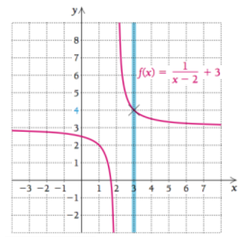
$\Rightarrow \lim_{x \to 3^+} f(x) = 4$

Thus, $\lim_{x \to 3} f(x) = 4.$




## 1.1 Limits: A Numerical and Graphical Approach

a) **Limit Graphically**




Observe on the graph that:

1) $\lim_{x \to 3^-} f(x) = 4$

and

2) $\lim_{x \to 3^+} f(x) = 4$

Therefore,

$\lim_{x \to 3} f(x) = 4.$




## 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically**

Let $x$ approach 2 from the left and right:

| $x \to 2^-$ | 1.5 | 1.9 | 1.99 | 1.999 |
|---|---|---|---|---|
| $f(x)$ | 1 | –7 | –97 | –997 |

$\Rightarrow \lim_{x \to 2^-} f(x) = -\infty$

| $x \to 2^+$ | 2.5 | 2.1 | 2.01 | 2.001 |
|---|---|---|---|---|
| $f(x)$ | 5 | 13 | 103 | 1003 |

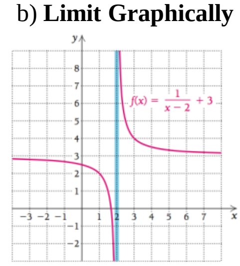
$\Rightarrow \lim_{x \to 2^+} f(x) = \infty$

Thus, $\lim_{x \to 2} f(x)$ does not exist.




## 1.1 Limits: A Numerical and Graphical Approach

b) **Limit Graphically**




Observe on the graph that:

1) $\lim_{x \to 2^-} f(x) = -\infty$

and

2) $\lim_{x \to 2^+} f(x) = \infty$

Therefore,

$\lim_{x \to 2} f(x)$ does not exist.

## 1.1 Limits: A Numerical and Graphical Approach

**Example 3:** Consider again the function $f$ given by

$$f(x) = \frac{1}{x-2} + 3$$

Find $\lim_{x \to \infty} f(x)$.




## 1.1 Limits: A Numerical and Graphical Approach

**Limit Numerically**

Note that you can only approach $\infty$ from the left:

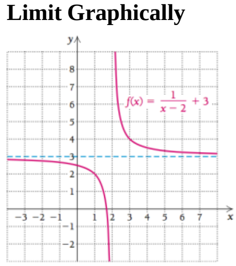
| $x \to \infty$ | 5 | 10 | 100 | 1000 |
|---|---|---|---|---|
| $f(x)$ | $3.\bar{3}$ | 3.125 | 3.0102 | 3.001 |

Thus, $\lim_{x \to \infty} f(x) = 3.$




## 1.1 Limits: A Numerical and Graphical Approach

**Limit Graphically**




Observe on the graph that, again, you can only approach $\infty$ from the left.

Therefore,

$$\lim_{x \to \infty} f(x) = 3.$$




## 1.1 Limits: A Numerical and Graphical Approach

Quick Check 3

Let $h(x) = \frac{1}{1-x} + 6.$ Find the following limits:

a.) $\lim_{x \to 1} h(x)$

b.) $\lim_{x \to 2} h(x)$

c.) $\lim_{x \to \infty} h(x)$

## 1.1 Limits: A Numerical and Graphical Approach

Quick Check 3 Solution

a.) $\lim_{x \to 1} h(x)$: Find the left-hand and right-hand limits as $x$ approaches $1$:

Left-hand Limit

| $x \to 1^-$ | $h(x)$ |
|---|---|
| 0 | 7 |
| 0.5 | 8 |
| 0.9 | 16 |
| 0.99 | 106 |
| 0.999 | 1006 |

Right-hand Limit

| $x \to 1^+$ | $h(x)$ |
|---|---|
| 2 | 5 |
| 1.5 | 4 |
| 1.1 | $-4$ |
| 1.01 | $-94$ |
| 1.001 | $-994$ |

Since the Left-Hand Limit goes to $\infty$ and the Right-Side Limit goes to $-\infty$,

the $\lim_{x \to 1} h(x) =$ does not exist.




## 1.1 Limits: A Numerical and Graphical Approach

Quick Check Solution Continued

b.) $\lim_{x \to 2} h(x)$: Find both the left-hand and right-hand limits as $x$ approaches $2$.

Left-Hand Limit

| $x \to 2^-$ | $h(x)$ |
|---|---|
| 1.1 | $-4$ |
| 1.5 | 4 |
| 1.9 | $4.\overline{8}$ |
| 1.99 | $4.\overline{98}$ |
| 1.999 | $4.\overline{998}$ |

Right-Hand Limit

| $x \to 2^+$ | $h(x)$ |
|---|---|
| 3 | 5.5 |
| 2.5 | $5.\overline{3}$ |
| 2.1 | $5.\overline{09}$ |
| 2.01 | $5.\overline{0099}$ |
| 2.001 | $5.\overline{000999}$ |

Since both the Left-Side Limit and Right-Side Limit agree, the $\lim_{x \to 2} h(x) = 5$.




## 1.1 Limits: A Numerical and Graphical Approach

Quick Check Solution Concluded

c.) $\lim_{x \to \infty} h(x)$: Find the limit as $x$ approaches $\infty$:

| $x \to \infty$ | $h(x)$ |
|---|---|
| 5 | 5.75 |
| 10 | $5.\overline{8}$ |
| 100 | $5.\overline{98}$ |
| 1000 | $5.\overline{998}$ |

Since both the Left-Side Limit and Right-Side Limit agree, the $\lim_{x \to \infty} h(x) = 6$.




## 1.1 Limits: A Numerical and Graphical Approach

### Section Summary

- The *limit* of a function $f$, as $x$ approaches $a$, is written $\lim_{x \to a} f(x) = L$.

  This means that as the values of $x$ approach $a$ the corresponding values of $f(x)$ approach $L$. The value of $L$ must be a unique, finite number.

- A *left-hand limit* is written $\lim_{x \to a^-} f(x)$.

  The values of $x$ are approaching $a$ from the left, that is, $x < a$.

- A *right-hand limit* is written $\lim_{x \to a^+} f(x)$.

  The values of $x$ are approaching $a$ from the right, that is, $x > a$.

- If the left-hand and right-hand limits (as $x$ approaches $a$) are *not* equal, the limit does *not* exist. On the other hand, if the left-hand and right-hand limits are equal, the limit does exist.
- A limit $\lim_{x \to a} f(x)$ may exist even though the function value $f(a)$ does not. (See Example 1.)
- A limit $\lim_{x \to a} f(x)$ may exist and be different from the function value $f(a)$. (See Example 3b.)
- Graphs and tables are useful tools in determining limits.