Algebraic Limits and Continuity

1.2

OBJECTIVE

- Develop and use the Limit Principles to calculate limits.
- Determine whether a function is continuous at a point.

1.2 Algebraic Limits and Continuity

LIMIT PROPERTIES:

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

we have the following:

L.1 $\lim_{x \to a} f(x) = f(a)$

if the function is *continuous* at *a*.

1.2 Algebraic Limits and Continuity

LIMIT PROPERTIES (continued):

L.2 The limit of a power is the power of the limit, and the limit of a root is the root of the limit.

That is, for any positive integer *n*,

$$\lim_{x \to a} f(x)^{n} = \left[\lim_{x \to a} f(x)\right]^n = L^n,$$

and

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L},$$

assuming that $L \geq 0$ when n is even.

1.2 Algebraic Limits and Continuity

LIMIT PROPERTIES (continued):

L.3 The limit of a sum or difference is the sum or difference of the limits.

$$\lim_{x \to a} \left[f(x) \pm g(x)\right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M.$$

L.4 The limit of a product is the product of the limits.

$$\lim_{x \to a} \left[f(x) \bullet g(x)\right] = \left[\lim_{x \to a} f(x)\right] \bullet \left[\lim_{x \to a} g(x)\right] = L \bullet M.$$

1.2 Algebraic Limits and Continuity

LIMIT PROPERTIES (concluded):

L.5 The limit of a quotient is the quotient of the limits.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{L}{M},\quad M\neq 0.$$

L.6 The limit of a constant times a function is the constant times the limit.

$$\lim_{x\to a} cf(x) = c\bullet\lim_{x\to a} f(x)=cL.$$

1.2 Algebraic Limits and Continuity

Example 1: Use the limit properties to find

$$\lim_{x\to 4}\left(x^2-3x+7\right)$$

The function is a parabola. It is continuous everywhere.

By Limit Property L1,

$$\lim_{x\to 4} x^2-3x+7 = 16-12+7=11$$

1.2 Algebraic Limits and Continuity

Example 2: Find $\lim_{x\to 0}\sqrt{\left(x^2-3x+2\right)}$

Limit Property L2 tell us

$$\lim_{x\to 0}\sqrt{x^2-3x+2} = \sqrt{\lim_{x\to 0}x^2-3x+2}=\sqrt{2}$$

1.2 Algebraic Limits and Continuity

Example 3: Find $\lim_{x\to -3}\frac{x^2-9}{x+3}$

Note –3 is not in the domain of $\frac{x^2-9}{x+3}$.

If $f(x)=g(x),\ \lim_{x\to c} f(x)=\lim_{x\to c} g(x)$

However, if we simplify $\frac{x^2-9}{x+3}$ first, the result can be evaluated at x = –3.

Example 3 (concluded):

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3} = \lim_{x \to -3} \frac{(x+3)(x-3)}{x+3}$$
$$= \lim_{x \to -3} x - 3$$
$$= -3 - 3$$
$$= -6$$

DEFINITION:

A function *f* is **continuous** at $x = a$ if:

1) $f(a)$ exists, (The output at *a* exists.)

2) $\lim_{x \to a} f(x)$ exists, (The limit as $x \to a$ exists.)

3) $\lim_{x \to a} f(x) = f(a)$. (The limit is the same as the output.)

A function is **continuous over an interval** if it is continuous at each point in that interval.

Example 4: Is the function *g* given by

$$g(x) = \begin{cases} \frac{1}{2}x + 3, & \text{for } x < -2 \\ x - 1, & \text{for } x \geq -2 \end{cases}$$

continuous at $x = -2$? Why or why not?

Note: the limit from the left does not equal the limit from the right; thus, limit does not exist at -2. -2 is a point of discontinuity.

Let $h(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & \text{for } x \neq 3 \\ 7, & \text{for } x = 3 \end{cases}$ Is h continuous at $x = 3$? Why or why not?

In order for $h(x)$ to continuous, $\lim_{x \to 3} h\ x\ = h\ 3$. So lets start by finding $\lim_{x \to 3} h\ x$.

$$\lim_{x \to 3} h\ x\ = \lim_{x \to 3} \frac{x^2 - 9}{x - 3}\ = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3}\ = \lim_{x \to 3} x + 3\ = 6$$

So the $\lim_{x \to 3} h\ x\ = 6$. However, $h\ 3\ = 7$, and thus $\lim_{x \to 3} h\ x\ \neq h\ 3$. Therefore $h\ x$ is not continuous at $x = 3$.

1.2 Algebraic Limits and Continuity

Let $p(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & \text{for } x \neq 5 \\ c, & \text{for } x = 5 \end{cases}$ Determine c such that p is continuous at $x = 5$.

In order for p to be continuous at $x = 5$, $\lim_{x \to 5} p(x) = p(5) = c$. So if we find $\lim_{x \to 5} p(x)$, we can determine what c is. Let's find $\lim_{x \to 5} p(x)$:

$$\lim_{x \to 5} p(x) = \lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x+5)}{x - 5} = \lim_{x \to 5} (x + 5) = 10$$

So $\lim_{x \to 5} p(x) = 10$. Therefore, in order for p to be continuous at $x = 5$, $c = 10$.