Average Rates of Change

1.3

OBJECTIVE

- Compute an average rate of change.
- Find a simplified difference quotient.

1.3 Average Rates of Change

DEFINITION:

The **average rate of change of *y* with respect to *x***, as x changes from x_1 to x_2, is the ratio of the change in output to the change in input:

$$\frac{y_2 - y_1}{x_2 - x_1}, \quad \text{where } x_2 \neq x_1.$$

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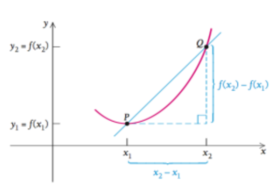
DEFINITION (concluded):

If we look at a graph of the function, we see that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

which is both the average rate of change *and* the slope of the line from $P(x_1, y_1)$ to $Q(x_2, y_2)$.



The line through P and Q, $\overleftrightarrow{PQ}$, is called a **secant line**.

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Example 1: A car finishes a race. Its distance in feet from the finish line after crossing it x seconds earlier is

$f(x) = 160x - 5x^2, 0 \le x \le 16$

Find the average rate in change of the function (average velocity) of the car from

a) $x = 8$ seconds to 12 seconds

b) $x = 8$ seconds to 9 seconds

c) $x = 8$ seconds to 8.5 seconds

d) $x = 8$ seconds to 8.1 seconds

e) $x = 8$ seconds to 8.001 seconds

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a) $\dfrac{f(12)-f(8)}{12-8}=\dfrac{1200-960}{4}=60\ ft/\sec\approx 41\text{mph}$

b) $\dfrac{f(9)-f(8)}{9-8}=\dfrac{1035-960}{1}=75\ ft/\sec\approx 51\text{mph}$

c) $\dfrac{f(8.5)-f(8)}{8.5-8}=\dfrac{998.75-960}{0.5}=77.5\ ft/\sec\approx 52.8\text{mph}$

d) $\dfrac{f(8.1)-f(8)}{8.1-8}=\dfrac{967.95-960}{0.1}=79.5\ ft/\sec\approx 54.2\text{mph}$

e) $\dfrac{f(8.001)-f(8)}{8.001-8}=\dfrac{960.079995-960}{0.001}=79.995\ ft/\sec\approx 54.5\text{mph}$

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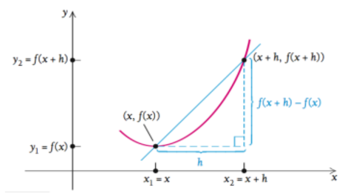
DEFINITION:

The average rate of change of f with respect to x is also called the **difference quotient**. It is given by

$$\frac{f(x+h)-f(x)}{h},\quad \text{where } h\neq 0.$$

The difference quotient is equal to the slope of the line from $(x, f(x))$ to $(x+h, f(x+h))$.



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Example 2: For $f(x)=160x-5x^2$ find the difference quotient when $x = 8$

$$\frac{f(x+h)-f(x)}{h}=\frac{f(8+h)-f(8)}{h}=$$

$$\frac{160(8+h)-5(8+h)^2-960}{h}=$$

$$\frac{1280+160h-5(64+16h+h^2)-960}{h}=$$

$$\frac{\cancel{1280}+160h-\cancel{320}-80h-5h^2-\cancel{960}}{h}=$$

$$\frac{80h-5h^2}{h}=\frac{\cancel{h}(80-5h)}{\cancel{h}}=80-5h$$

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Example 3: For $f(x)=x^3$ find a simplified form of the difference quotient.

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^3-x^3}{h}$$

$$=\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$$

$$=\frac{\cancel{h}\,3x^2+3xh+h^2}{\cancel{h}}$$

$$=3x^2+3xh+h^2,\quad h\neq 0.$$

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Quick Check 3

Use the result of Example 3 to calculate the slope of the secant line (average rate of change) at $x = 2$, for $h = 0.1$, $h = 0.01$, and $h = 0.001$.

Using the formula found in Example 6 ($3x^2 + 3xh + h^2, h \neq 0$)

For $h = 0.1$: $3(2)^2 + 3(2)(0.1) + 0.1^2 = 12 + 0.6 + 0.01 = 12.61$

For $h = 0.01$: $3(2)^2 + 3(2)(0.01) + 0.01^2 = 12 + 0.06 + 0.0001 = 12.0601$

For $h = 0.001$: $3(2)^2 + 3(2)(0.001) + 0.001^2 = 12 + 0.006 + 0.000001$

$= 12.006001$

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Example 4: For $f(x) = \dfrac{3}{x}$ find a simplified form of the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{3}{x+h} - \dfrac{3}{x}}{h} = \frac{\dfrac{3x - 3(x+h)}{x(x+h)}}{h}$$

$$= \frac{\dfrac{\cancel{3x} - \cancel{3x} - 3h}{x(x+h)}}{h} = \frac{\dfrac{-3\cancel{h}}{x(x+h)}}{\cancel{h}}$$

$$= \frac{-3}{x(x+h)}, \quad h \neq 0.$$