Differentiation Using Limits of Difference Quotients

1.4

OBJECTIVE

- Find derivatives and values of derivatives
- Find equations of tangent lines

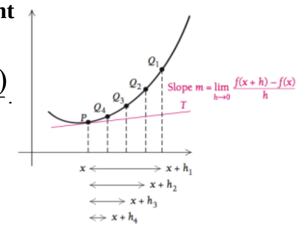
1.4 Differentiation Using Limits of Difference Quotients

DEFINITION:

The **slope of the tangent line** at (x, $f(x)$) is

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This limit is also the **instantaneous rate of change** of $f(x)$ at x.



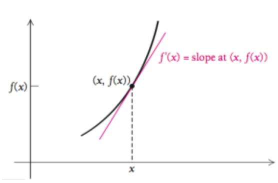
1.4 Differentiation Using Limits of Difference Quotients

DEFINITION:

For a function $y = f(x)$, its **derivative** at x is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. If $f'(x)$ exists, then we say that f is **differentiable** at x.



1.4 Differentiation Using Limits of Difference Quotients

Example 1: For $f(x) = x^2$, find $f'(x)$ Then find $f'(-3)$ and $f'(4)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$f'(x) = \lim_{h \to 0} (2x + h)$$

$$f'(x) = 2x$$

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Example 1 (concluded):

$$f'(x) = 2x$$

$$f'(-3) = 2(-3) = -6$$

$$f'(4) = 2(4) = 8$$

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Example 2: For $f(x) = x^3$, find $f'(x)$.
Then find $f'(-1)$ and $f'(1.5)$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} = \lim_{h\to 0} 3x^2 + 3xh + h^2$$

$$f'(x) = 3x^2$$

1.4 Differentiation Using Limits of Difference Quotients

Use the results from Examples 1 and 2 to find the derivative

$$f(x) = x^3 + x^2$$

$$f'(x) = 3x^2 + 2x$$

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Example 3: For $f(x) = \frac{3}{x}$:

a) Find $f'(x)$.

b) Find $f'(2)$.

c) Find an equation of the tangent line to the curve at $x = 2$.

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Example 3 (continued):

a) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{3}{x+h}-\frac{3}{x}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{3x-3(x+h)}{x(x+h)}}{h}=\lim_{h\to 0}\frac{\frac{3x-3x-3h}{x(x+h)}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{-3h}{x(x+h)}}{h}=\lim_{h\to 0}\frac{-3}{x(x+h)}=-\frac{3}{x^2}.$$

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Example 3 (continued):

b) $$f'(x)=-\frac{3}{x^2}$$

$$f'(2)=-\frac{3}{2^2}=-\frac{3}{4}$$

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Example 3 (concluded):

c) $x=2,\ m=f'(2)=-\frac{3}{4},\ y=f(2)=\frac{3}{2}$

$$y = mx+b$$

$$\frac{3}{2} = -\frac{3}{4}\cdot 2+b$$

$$\frac{3}{2} = -\frac{3}{2}+b$$

$$3 = b$$

Thus, $y = -\frac{3}{4}x+3$ is the equation of the tangent line.

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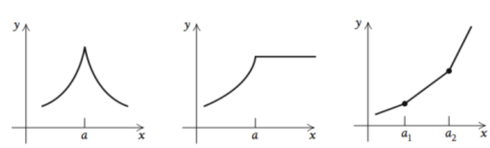
Where a Function is Not Differentiable:

1) A function $f(x)$ is not differentiable at a point $x = a$, if there is a "corner" at a.



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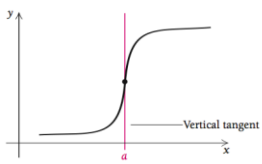
Where a Function is Not Differentiable:

2) A function $f(x)$ is not differentiable at a point $x = a$, if there is a vertical tangent at a.



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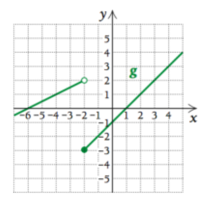
Where a Function is Not Differentiable:

3) A function $f(x)$ is not differentiable at a point $x = a$, if it is not continuous at a.

Example: $g(x)$ is not continuous at -2, so $g(x)$ is not differentiable at $x = -2$.



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Where is $f(x) = |x-6|$ not differentiable? Why?

$f(x) = |x-6|$ is not differentiable at $x = 6$. This is the vertex of the function, and is considered a corner of the function. Therefore, $f(x) = |x-6|$ is not differentiable at $x = 6$.