Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

2.1

OBJECTIVE

- Find relative extrema of a continuous function using the First-Derivative Test.
- Sketch graphs of continuous functions.

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM 1

If $f'(x) > 0$ for all x in an interval I, then f is increasing over I.

If $f'(x) < 0$ for all x in an interval I, then f is decreasing over I.

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DEFINITION:

A **critical value** of a function f is c *if*

$$f(c) \text{ exists AND}$$
$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

If $f(c)$ is a **relative minimum of maximum (relative extrema)** then c is a critical value.

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM 2: The First-Derivative Test for Relative Extrema

For any continuous function f that has exactly one critical value c in an open interval (a, b);

F1. f has a relative minimum at c if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b). That is, f is decreasing to the left of c and increasing to the right of c.

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THEOREM 3: The First-Derivative Test for Relative Extrema (continued)

F2. f has a relative maximum at c if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b). That is, f is increasing to the left of c and decreasing to the right of c.

F3. f has neither a relative maximum nor a relative minimum at c if $f'(x)$ has the same sign on (a, c) and (c, b).

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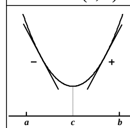
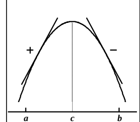
Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	Relative minimum	–	+	Decreasing on $(a, c]$; increasing on $[c, b)$
	Relative maximum	+	–	Increasing on $(a, c]$; decreasing on $[c, b)$

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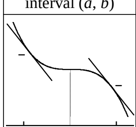
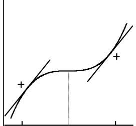
Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	No relative maxima or minima	–	–	Decreasing on (a, b)
	No relative maxima or minima	+	+	Increasing on (a, b)

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1: Graph the function f given by

$$f(x) = 2x^3 - 3x^2 - 12x + 12.$$

and find the relative extrema.

Suppose that we are trying to graph this function but do not know any calculus. What can we do? We can plot a few points to determine in which direction the graph seems to be turning. Let's pick some x-values and see what happens.

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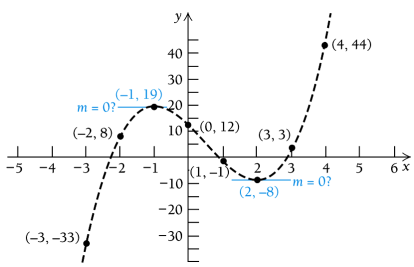
x	$f(x)$
−3	−33
−2	8
−1	19
0	12
1	−1
2	−8
3	3
4	44

Example 1 (continued):



2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

We can see some features of the graph from the sketch. Now we will calculate the coordinates of these features precisely.

1st find a general expression for the derivative.

$$f'(x) = 6x^2 - 6x - 12$$

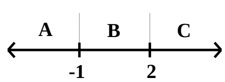
2nd determine where $f'(x)$ does not exist or where $f'(x) = 0$. (Since $f'(x)$ is a polynomial, there is no value where $f'(x)$ does not exist. So, the only possibilities for critical values are where $f'(x) = 0$.)

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

$$6x^2 - 6x - 12 = 0$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \quad or \quad x = -1$$

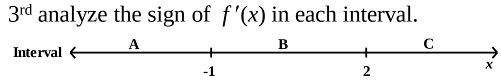
These two critical values partition the number line into 3 intervals: A (−∞, −1), B (−1, 2), and C (2, ∞).



2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

3rd analyze the sign of $f'(x)$ in each interval.



Test Value	$x = -2$	$x = 0$	$x = 4$
Sign of $f'(x)$	+	−	+
Result	f is increasing on (−∞, −1]	f is decreasing on [−1, 2]	f is increasing on [2, ∞)

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Example 1 (concluded):

Therefore, by the First-Derivative Test,

f has a relative maximum at $x = -1$ given by

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 12 = 19$$

Thus, (–1, 19) is a relative maximum.

And f has a relative minimum at $x = 2$ given by

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 12 = -8$$

Thus, (2, –8) is a relative minimum.

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example: Find the relative extrema for the Function $f(x)$ given by

$$f(x) = (x-2)^{2/3} + 1$$

Then sketch the graph.

1st find $f'(x)$.

$$f'(x) = \frac{2}{3}(x-2)^{-1/3}$$

$$f'(x) = \frac{2}{3\sqrt[3]{x-2}}$$

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Example 2 (continued):

Then, find where $f'(x)$ does not exist or where $f'(x) = 0$.

Note that $f'(x)$ does not exist where the denominator equals 0. Since the denominator equals 0 when $x = 2$, $x = 2$ is a critical value.

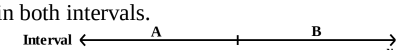
$f'(x) = 0$ where the numerator equals 0. Since $2 \neq 0$, $f'(x) = 0$ has no solution.

Thus, $x = 2$ is the only critical value.

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

3rd $x = 2$ partitions the number line into 2 intervals: A $(-\infty, 2)$ and B $(2, \infty)$. So, analyze the signs of $f'(x)$ in both intervals.



Test Value	$x = 0$	$x = 3$
Sign of $f'(x)$	–	+
Result	f is decreasing on $(-\infty, 2]$	f is increasing on $[2, \infty)$

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

Therefore, by the First-Derivative Test,

f has a relative minimum at $x = 2$ given by

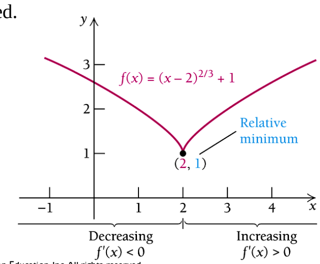
$$f(2) = (2-2)^{2/3} + 1 = 1$$

Thus, (2, 1) is a relative minimum.

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (concluded):

We use the information obtained to sketch the graph below, plotting other function values as needed.



x	$f(x)$, approximately
−1	3.08
−0.5	2.84
0	2.59
0.5	2.31
1	2
1.5	1.63
2	1
2.5	1.63
3	2
3.5	2.31
4	2.59

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 2

Find the relative extrema of the function h given by $h(x) = x^4 - \frac{8}{3}x^3$. Then sketch the graph.

First find $h'(x)$:

$$h'(x) = 4 \cdot x^3 - 3 \cdot \frac{8}{3}x^2$$

$$h'(x) = 4x^3 - 8x^2$$

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 2 Continued

Second find where $h'(x)$ does not exist or where $h'(x) = 0$.

$$h'(x) = 4x^3 - 8x^2 = 0$$

$$4x^2(x-2) = 0$$

So $h'(x) = 0$ when $x = 0$ and $x = 2$.

Thus $x = 0$ and $x = 2$ are the critical values.

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Quick Check 2 Continued

Third, $x=0$ and $x=2$ partitions the number line into three intervals: A $(-\infty,0)$, B $(0,2)$, and C $(2,\infty)$. So analyze the signs of $h'(x)$ in all three intervals.

Interval	A	B	C
Test Value	$x=-2$	$x=1$	$x=3$
Sign of $h'(x)$	–	–	+
Result	h is decreasing	h is decreasing	h is increasing

Thus, there is a minimum at $x=2$. Therefore, $\left(2,\frac{-16}{3}\right)$ is a minimum.

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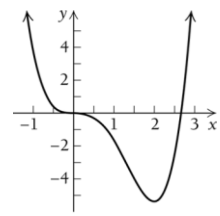
Quick Check 2 Concluded

From the information we have gathered, the graph of $h(x)$ looks like:



$h(x)=x^4-\frac{8}{3}x^3$