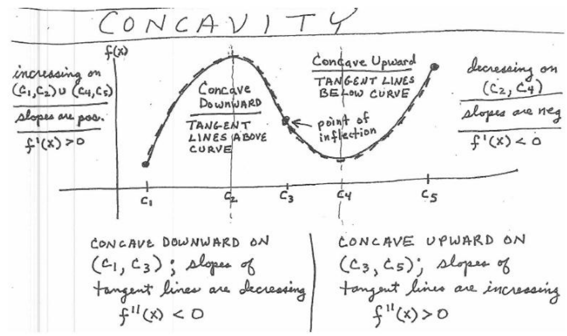
Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

2.2

OBJECTIVE

- Find the relative extrema of a function using the Second-Derivative Test.
- Sketch the graph of a continuous function.



2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM : A Test for Concavity

1. If $f''(x) > 0$ on an interval I, then the graph of f is concave up. (f' is increasing, so f is turning up on I.)

2. If $f''(x) < 0$ on an interval I, then the graph of f is concave down. (f' is decreasing, so f is turning down on I.)

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM : Finding Points of Inflection

If a function f has a point of inflection, it must occur at a point x_0, where.

$$f''(x_0) = 0 \quad \text{or} \quad f''(x_0) \text{ does not exist.}$$

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THEOREM : The Second Derivative Test for Relative Extrema

Suppose that f is differentiable for every x in an open interval (a, b) and that there is a critical value c in (a, b) for which $f'(c) = 0$. Then:

1. $f(c)$ is a relative minimum if $f''(c) > 0$.
2. $f(c)$ is a relative maximum if $f''(c) < 0$.

For $f''(c) = 0$, the First-Derivative Test can be used to determine whether $f(c)$ is a relative extremum.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1: Graph the function f given by

$$f(x) = x^3 + 3x^2 - 9x - 13,$$

and find the relative extrema.

1st find $f'(x)$ and $f''(x)$.

$$f'(x) = 3x^2 + 6x - 9,$$
$$f''(x) = 6x + 6.$$

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

2nd solve $f'(x) = 0$.

$$\begin{aligned} 3x^2 + 6x - 9 &= 0 \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \end{aligned}$$

$$x + 3 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -3 \qquad\quad x = 1$$

Thus, $x = -3$ and $x = 1$ are critical values.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (continued):

3rd use the Second Derivative Test with –3 and 1.

$f''(-3) = 6(-3) + 6 = -18 + 6 = -12 < 0$: Relative maximum

$f''(1) = 6(1) + 6 = 6 + 6 = 12 > 0$: Relative minimum

Lastly, find the values of $f(x)$ at –3 and 1.

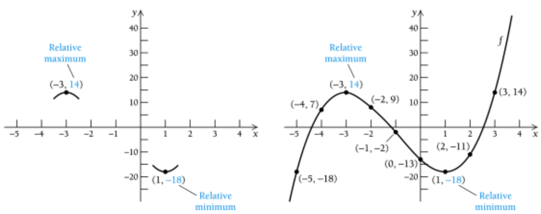
$$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 13 = 14$$
$$f(1) = (1)^3 + 3(1)^2 - 9(1) - 13 = -18$$

So, (–3, 14) is a relative maximum and (1, –18) is a relative minimum.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 1 (concluded):

Then, by calculating and plotting a few more points, we can make a sketch of $f(x)$, as shown below.



2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Strategy for Sketching Graphs:

a) *Derivatives and Domain*. Find $f'(x)$ and $f''(x)$. Note the domain of f.

b) *Critical values of f*. Find the critical values by solving $f'(x) = 0$ and finding where $f'(x)$ does not exist. Find the function values at these points.

c) *Increasing and/or decreasing; relative extrema*. Substitute each critical value, x_0, from step (b) into $f''(x)$ and apply the Second Derivative Test.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Strategy for Sketching Graphs (continued):

d) *Inflection Points*. Determine candidates for inflection points by finding where $f''(x) = 0$ or where $f''(x)$ does not exist. Find the function values at these points.

e) *Concavity*. Use the candidates for inflection points from step (d) to define intervals. Use the relative extrema from step (b) to determine where the graph is concave up and where it is concave down.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Strategy for Sketching Graphs (concluded):

f) *Sketch the graph*. Sketch the graph using the information from steps (a) – (e), calculating and plotting extra points as needed.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2: Find the relative extrema of the function f given by

$$f(x) = x^3 - 3x + 2,$$

and sketch the graph.

a) *Derivatives and Domain*.

$$f'(x) = 3x^2 - 3,$$
$$f''(x) = 6x.$$

The domain of f is all real numbers.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

b) Critical values of f.

$$3x^2 - 3 = 0$$
$$3x^2 = 3$$
$$x^2 = 1$$
$$x = \pm 1$$

And we have $f(-1) = 4$ and $f(1) = 0$.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

c) *Increasing and/or Decreasing; relative extrema.*

$$f''(-1) = 6(-1) = -6 < 0$$

So (–1, 4) is a relative maximum, and $f(x)$ is increasing on $(-\infty, -1]$ and decreasing on $[-1, 1]$. The graph is also concave down at the point (–1, 4).

$$f''(1) = 6(1) = 6 > 0$$

So (1, 0) is a relative minimum, and $f(x)$ is decreasing on $[-1, 1]$ and increasing on $[1, \infty)$. The graph is also concave up at the point (1, 0).

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (continued):

d) *Inflection Points.*

$$6x = 0$$
$$x = 0$$

And we have $f(0) = 2$.

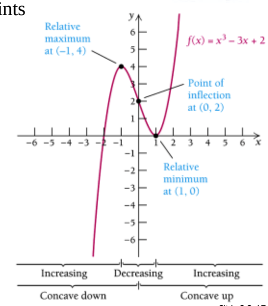
e) *Concavity.* From step (c), we can conclude that f is concave down on the interval $(-\infty, 0)$ and concave up on $(0, \infty)$.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Example 2 (concluded)

f) *Sketch the graph*. Using the points from steps (a) – (e), the graph follows.

x	$f(x)$
−3	−16
−2	0
−1	4
0	2
1	0
2	4
3	20



2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Section Summary

- The second derivative f'' determines the ***concavity*** of the graph of function f.
- If $f''\ x > 0$ for all x in an open interval I, then the graph of f is ***concave up*** over I.
- If $f''\ x < 0$ for all x in an open interval I, then the graph of f is ***concave down*** over I.
- If c is a critical value and $f''\ c > 0$, then $f(c)$ is a relative minimum.
- If c is a critical value and $f''\ c > 0$, then $f(c)$ is a relative maximum.

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

Section Summary Concluded

- If c is a critical value and $f''\ c = 0$, the First-Derivative Test must be used to classify $f(c)$.
- If $f''(x_0) = 0$ or $f''(x_0)$ does not exist, and there is a change in concavity to the left and to the right of x_0, then the point $(x_0, f(x_0))$ is called a ***point of inflection***.
- Finding the extrema, intervals over which a function is increasing or decreasing, intervals of upward or downward concavity, and points of inflection is all part of a strategy for accurate curve sketching.