Graph Sketching: Asymptotes and Rational Functions

2.3

OBJECTIVE

- Find limits involving infinity.
- Determine the asymptotes of a function's graph.
- Graph rational functions.

2.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

A **rational function** is a function f that can be described by

$$f(x)=\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials, with $Q(x)$ not the zero polynomial. The domain of f consists of all inputs x for which $Q(x)\neq 0$.

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DEFINITION:

The line $x = a$ is a **vertical asymptote** if any of the following limit statements are true:

$$\lim_{x\to a^-} f(x)=\infty \quad \text{or} \quad \lim_{x\to a^-} f(x)=-\infty \quad \text{or}$$

$$\lim_{x\to a^+} f(x)=\infty \quad \text{or} \quad \lim_{x\to a^+} f(x)=-\infty.$$

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DEFINITION:

The line $y = b$ is a **horizontal asymptote** if either or both of the following limit statements are true:

$$\lim_{x\to -\infty} f(x)=b \quad \text{or} \quad \lim_{x\to \infty} f(x)=b.$$

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Quick Check 2 Concluded

Since both the numerator and denominator have the same power of x, we can divide both by that power:

$$f(x)=\frac{2x^2+x-1}{15x^2+28x+12}=\frac{2+\frac{1}{x}-\frac{1}{x^2}}{15+\frac{28}{x}+\frac{12}{x^2}}$$

Now we can see that as $|x|$ gets very large, the numerator approaches 2 and the denominator approaches 15. Therefore the value of the function gets very close to $\frac{2}{15}$. Thus, $\lim_{x\to-\infty} f(x)=\frac{2}{15}$ and $\lim_{x\to\infty} f(x)=\frac{2}{15}$.

Therefore there is a horizontal asymptote at $y=\frac{2}{15}$.

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DEFINITION:

A linear asymptote that is neither vertical nor horizontal is called a **slant**, or **oblique**, **asymptote**. For any rational function of the form $f(x) = p(x)/q(x)$, a slant asymptote occurs when the degree of $p(x)$ is exactly 1 more than the degree of $q(x)$. A graph can cross a slant asymptote.

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Example 3: Find the slant asymptote:

$$f(x)=\frac{x^2-4}{x-1}$$

First, divide the numerator by the denominator.

$$\begin{array}{r} x+1 \\ x-1\overline{)x^2 \quad -4} \\ \underline{x^2-x} \\ x-4 \\ \underline{x-1} \\ -3 \end{array}$$

$$\Rightarrow \quad f(x)=\frac{x^2-4}{x-1}=(x+1)+\frac{-3}{x-1}$$

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Example 3 (concluded):

Second, now we can see that as $|x|$ gets very large, $-3/(x-1)$ approaches 0. Thus, for very large $|x|$, the expression $x + 1$ is the dominant part of

$$(x+1)+\frac{-3}{x-1}$$

thus $y = x + 1$ is the slant asymptote.