Implicit Differentiation and Related Rates

2.7

OBJECTIVE

- Differentiate implicitly.
- Solve related rate problems.

2.7 Implicit Differentiation and Related Rates

To differentiate implicitly :

a) Differentiate both sides of the equation with respect to x (or whatever variable you are differentiating with respect to).

b) Apply the rules of differentiation as necessary. Any time an expression involving y is differentiated, dy/dx will be a factor in the result.

c) Isolate all terms with dy/dx as a factor on one side of the equation.

d) Factor out dy/dx.

e) Divide both sides of the equation to isolate dy/dx.

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Example 1: For $y^3 + x^2y^5 - x^4 = 27$

a) Find dy/dx using implicit differentiation.

b) Find the slope of the tangent line to the curve at the point (0, 3).

a)
$$\frac{d}{dx}\left(y^3 + x^2y^5 - x^4\right) = \frac{d}{dx}(27)$$
$$3y^2 \cdot \frac{dy}{dx} + x^2 \cdot 5y^4 \cdot \frac{dy}{dx} + y^5 \cdot 2x - 4x^3 = 0$$
$$\frac{dy}{dx}\left(3y^2 + 5x^2y^4\right) = 4x^3 - 2xy^5$$

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Example 1 (concluded):

$$\frac{dy}{dx} = \frac{4x^3 - 2xy^5}{3y^2 + 5x^2y^4}$$

b) The slope of the tangent line at (0, 3) is dy/dx at (0, 3).

$$\frac{dy}{dx} = \frac{4(0)^3 - 2(0)(3)^5}{3(3)^2 + 5(0)^2(3)^4}$$
$$\frac{dy}{dx} = \frac{0}{27}$$
$$\frac{dy}{dx} = 0$$

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Quick Check 1

For $y^2x+2x^3y^3=y+1$, find dy/dx using implicit differentiation.

$$\frac{d}{dx}(y^2x+2x^3y^3)=\frac{d}{dx}(y+1)$$

$$2xy\frac{dy}{dx}+y^2+6x^3y^2\frac{dy}{dx}+6x^2y^3=1\frac{dy}{dx}$$

$$y^2+6x^2y^3=1\frac{dy}{dx}-2xy\frac{dy}{dx}-6x^3y^2\frac{dy}{dx}$$

$$y^2+6x^2y^3=(1-2xy-6x^3y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{y^2+6x^2y^3}{1-2xy-6x^3y^2}$$

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Example 2: For the demand equation $x=\sqrt{200-p^3}$ differentiate implicitly to find dp/dx.

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\sqrt{200-p^3}\right)$$

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(200-p^3\right)^{1/2}$$

$$1=\frac{1}{2}\cdot(200-p^3)^{-1/2}\cdot\left(-3p^2\right)\cdot\frac{dp}{dx}$$

$$1=\frac{-3p^2}{2\sqrt{200-p^3}}\cdot\frac{dp}{dx}\quad\Rightarrow\quad\frac{dp}{dx}=\frac{2\sqrt{200-p^3}}{-3p^2}$$

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Example 3: A restaurant supplier services the restaurants in a circular area in such a way that the radius r is increasing at the rate of 2 mi. per year at the moment when $r = 5$ mi. At that moment, how fast is the area increasing?

$$A=\pi r^2$$

$$\frac{dA}{dr}=2\pi r\cdot\frac{dr}{dt}$$

$$\frac{dA}{dr}=2\pi\cdot 5\text{ mi}\cdot 2\ \frac{\text{mi}}{\text{yr}}=20\pi\ \frac{\text{mi}^2}{yr}\approx 63\text{ sq mi per yr.}$$

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Quick Check 2

A spherical balloon is deflating, losing 20 cm^3 of air per minute. At the moment when the radius of the balloon is 8 cm, how fast is the radius decreasing (*Hint:* $V=\frac{4}{3}\pi r^3$)?

$$\frac{dV}{dr}=4\pi r^2\frac{dr}{dt}$$

$$20\text{ cm}^3=4\pi(8\text{ cm})^2\frac{dr}{dt}$$

$$\frac{dr}{dt}=-\frac{20\text{ cm}^3}{4\pi\left(64\text{ cm}^2\right)}=-\frac{5\text{ cm}^3}{64\ \pi\text{cm}^2}\approx -0.025\text{ cm}$$

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Example 4: For Luce Landscaping, the total revenue from the yard maintenance of x homes is given by

$$R(x) = 1000x - x^2,$$

and the total cost is given by

$$C(x) = 3000 + 20x.$$

Suppose that Luce is maintaining 10 homes per day at the moment when the 400th customer is signed. At that moment, what is the rate of change of (a) total revenue, (b) total cost, and (c) total profit?

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Example 4 (continued):

a) $$\frac{dR}{dt} = 1000 \cdot \frac{dx}{dt} - 2x \cdot \frac{dx}{dt}$$

$$\frac{dR}{dt} = 1000 \cdot 10 - 2 \cdot 400 \cdot 10 = \$2000 / \text{day}.$$

b) $$\frac{dC}{dt} = 20 \cdot \frac{dx}{dt}$$

$$\frac{dC}{dt} = 20 \cdot 10 = \$200 / \text{day}.$$

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Example 4 (concluded):

c) Since $P = R - C$

$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt}$$

$$\frac{dP}{dt} = \$2000 / \text{day} - \$200 / \text{day}$$

$$\frac{dP}{dt} = \$1800 / \text{day}.$$

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Section Summary

- If variables x and y are related to one another by an equation but neither variable is isolated on one side of the equation, we say that x and y have an implicit relationship. To find dy/dx without solving such an equation for y, we use *implicit differentiation*.
- Whenever we implicitly differentiate y with respect to x, the factor dy/dx will appear as a result of the Chain Rule.
- To determine the slope of a tangent line at a point on the graph of an implicit relationship, we may need to evaluate the derivative by inserting both the $x-$value and the $y-$value of the point of tangency.