3.1 Exponential Functions

OBJECTIVE

- Graph exponential functions.
- Differentiate exponential functions.

3.1 Exponential Functions

DEFINITION:

An **exponential function** f is given by

$$f(x) = a^x,$$

where x is any real number, $a > 0$, and $a \neq 1$. The number a is called the **base**.

Examples: $f(x) = 2^x$, $f(x) = \left(\frac{1}{2}\right)^x$, $f(x) = (0.4)^x$

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Example 1: Graph $f(x) = 2^x$. First, we find some function values.

x	$y = f(x) = 2^x$
0	1
$\frac{1}{2}$	1.4
1	2
2	4
3	8
−1	$\frac{1}{2}$
−2	$\frac{1}{4}$

$x = 0$, $y = 2^0 = 1$;

$x = \frac{1}{2}$, $y = 2^{1/2} = \sqrt{2} \approx 1.4$;

$x = 1$, $y = 2^1 = 2$;

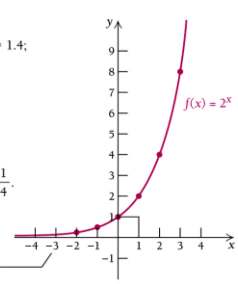
$x = 2$, $y = 2^2 = 4$;

$x = 3$, $y = 2^3 = 8$;

$x = -1$, $y = 2^{-1} = \frac{1}{2}$;

$x = -2$, $y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$.

The curve comes very close to the x-axis, but does not touch or cross it. The x-axis is a horizontal asymptote.



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DEFINITION:

$$e = \lim_{h \to 0} (1 + h)^{1/h} \approx 2.718281828459$$

We call e the *natural base*.

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THEOREM

The derivative of the function *f* given by $f(x) = e^x$ is itself:

$$f'(x) = f(x), \quad \text{or} \quad \frac{d}{dx}e^x = e^x$$

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Example: Find *dy/dx*: $y = x^2e^x$

$$\frac{d}{dx}\left(x^2e^x\right) = x^2 \cdot e^x + e^x \cdot 2x$$

$$= e^x\left(x^2 + 2x\right)$$

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THEOREM

$$\frac{d}{dx}e^u = e^u \cdot u'$$

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Example: Differentiate each of the following with respect to *x*:

$$y = e^{-x^2+4x-7}$$

$$\frac{d}{dx}e^{-x^2+4x-7} = e^{-x^2+4x-7} \cdot \left(-2x + 4\right)$$

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Example : Graph $h(x) = 1 - e^{-2x}$ with $x \geq 0$. Analyze the graph using calculus.

First, we find some values, plot the points, and sketch the graph.

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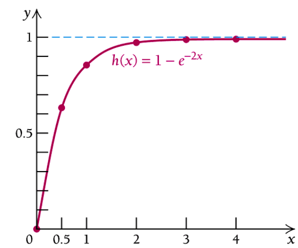
Example (continued):

x	$h(x)$
0	0
0.5	0.63212
1	0.86466
2	0.98168
3	0.99752
4	0.99966
5	0.99995



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Example 4 (continued):

a) *Derivatives.* Since $h(x) = 1 - e^{-2x}$,

$$h'(x) = 2e^{-2x}$$

and

$$h''(x) = -4e^{-2x}.$$

b) *Critical values*. Since $e^{-2x} = \frac{1}{e^{2x}} > 0$, the derivative $h'(x) = 2e^{-2x} > 0$ for all real numbers *x*. Thus, the derivative exists for all real numbers, and the equation $h'(x) = 0$ has no solution. There are no critical values.

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Example (continued):

c) *Increasing.* Since the derivative $h'(x) = 2e^{-2x} > 0$ for all real numbers *x*, we know that *h* is increasing over the entire real number line.

d) *Inflection Points.* Since $h''(x) = -4e^{-2x} > 0$ we know that the equation $h''(x) = 0$ has no solution. Thus there are no points of inflection.

Example (concluded):

e) *Concavity.* Since $h''(x) = -4e^{-2x} > 0$ for all real numbers *x*, we know that h' is decreasing and the graph is concave down over the entire real number line.