Logarithmic Functions

3.2

OBJECTIVE

- Convert between logarithmic and exponential equations.
- Solve exponential equations.
- Solve problems involving exponential and logarithmic functions.
- Differentiate functions involving natural logarithms.

DEFINITION:

A **logarithm** is defined as follows:

$$\log_a x = y \quad \text{means} \quad a^y = x, \ \ a > 0, \ \ a \neq 1.$$

The number $\log_a x$ is the power y to which we raise a to get x. The number a is called the *logarithmic base*. We read $\log_a x$ as "the logarithm, base a, of x."

Example: Graph: $y = \log_2 x$.

First we write the equivalent exponential equation:

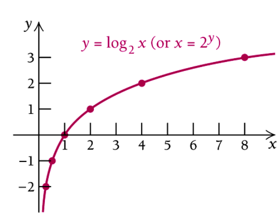
$$2^y = x$$

Then, we will select some values of y to find the corresponding values of x. Next, we will plot these points, keeping in mind that x is still the first coordinate, and connect the points with a smooth curve.

Example (concluded):

x, or 2^y	y
1	0
2	1
4	2
8	3
$\frac{1}{2}$	−1
$\frac{1}{4}$	−2

① Select y.
② Compute x.



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THEOREM: Properties of Logarithms

For any positive numbers M, N, a, and b, $b \neq 1$, and any real number k:

P1. $\log_a(MN) = \log_a M + \log_a N$

P2. $\log_a \frac{M}{N} = \log_a M - \log_a N$

P3. $\log_a(M^k) = k \cdot \log_a M$

P4. $\log_a a = 1$

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THEOREM (concluded):

P5. $\log_a(a^k) = k$

P6. $\log_a 1 = 0$

P7. $\log_b M = \frac{\log_a M}{\log_a b}$

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Example 2: Given $\log_a 2 = 0.301$ and $\log_a 3 = 0.477$ find each of the following:

a) $\log_a 6$;

b) $\log_a \frac{2}{3}$;

c) $\log_a 81$;

d) $\log_a \frac{1}{3}$;

a) $$\log_a 6 = \log_a(2 \cdot 3) = \log_a 2 + \log_a 3 = 0.301 + 0.477 = 0.778$$

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Example 2 (continued):

b) $$\log_a \frac{2}{3} = \log_a 2 - \log_a 3 = 0.301 - 0.477 = -0.176$$

c) $$\log_a 81 = \log_a 3^4 = 4\log_a 3 = 4(0.477) = 1.908$$

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d) $\log_a \frac{1}{3}$ $\quad \log_a 1 - \log_a 3$

$= \quad 0 - 0.477$

$= \quad -0.477$

3.2 Logarithmic Functions

DEFINITION:

For any positive number x, $\log x = \log_{10} x$.

3.2 Logarithmic Functions

DEFINITION:

For any positive number x, $\ln x = \log_e x$.

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Example 3: Solve $e^{-0.04t} = 0.05$ for t.

$$\begin{aligned} \ln e^{-0.04t} &= \ln 0.05 \\ -0.04t &= \ln 0.05 \\ t &= \frac{\ln 0.05}{-0.04} \\ t &= 75 \end{aligned}$$

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THEOREM

For any positive number *x*,

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

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Differentiate: $y = \frac{\ln x}{x^2}$

$$y' = \frac{x^2\left(\frac{1}{x}\right) - 2x\ln x}{x^4} = \frac{\frac{x^2}{x} - 2x\ln x}{x^4}$$

$$= \frac{x - 2x\ln x}{x^4} = \frac{x\;1 - 2\ln x}{x^4}$$

$$= \frac{1 - 2\ln x}{x^3}$$

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THEOREM

$$\frac{d}{dx}\ln f(x) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)},$$

or

$$\frac{d}{dx}\ln u = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}$$

The derivative of the natural logarithm of a function is derivative of the function divided by the function.

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Example: Differentiate $y = \ln(x^2 - 5)$

$$\frac{dy}{dx} = \frac{2x}{x^2 - 5}.$$

Example: Differentiate $f(x) = \ln\left(\frac{x^3+4}{x}\right)$.

$$f'(x) = \frac{d}{dx}[\ln(x^3+4) - \ln x]$$

$$= \frac{3x^2}{x^3+4} - \frac{1}{x}$$