Antidifferentiation

4.1

OBJECTIVE

- Find an antiderivative of a function.
- Evaluate indefinite integrals using the basic integration formulas.
- Use initial conditions, or boundary conditions, to determine an antiderivative.

4.1 Antidifferentiation

THEOREM

The **antiderivative** of $f(x)$ is the set of functions $F(x)+C$ such that

$$\frac{d}{dx}\left[F(x)+C\right]=f(x).$$

The constant C is called the **constant of integration.**

If $f(x)=3x^2$ then

$F(x)=x^3+2$

$F(x)=x^3-11$

$F(x)=x^3+\pi$

In general, $\int 3x^2 dx = x^3 + C$

4.1 Antidifferentiation

Integrals and Integration

Finding the antiderivative is called integrating.

To indicate the antiderivative of $3x^2$ is $x^3 + C$, we write $\int x^2 dx = x^3 + C$, where $\int f(x)dx$ is called the indefinite integral.

$\int f(x)dx = F(x) + C$, is the general form.

4.1 Antidifferentiation

Example: Determine these indefinite integrals. That is, find the antiderivative of each integrand:

a.) $\int 8dx = 8x + C$ Check: $\frac{d}{dx}(8x+C)=8$

b.) $\int 3x^2 dx = x^3 + C$ Check: $\frac{d}{dx}(x^3+C)=3x^2$

c.) $\int e^x dx = e^x + C$ Check: $\frac{d}{dx}(e^x+C)=e^x$

d.) $\int \frac{1}{x} dx = \ln|x| + C$ Check: $\frac{d}{dx}(\ln x + C)=\frac{1}{x}$

4.1 Antidifferentiation

THEOREM: Basic Integration Formulas

1. $\int k\, dx = kx + C \quad (k \text{ is a constant})$

2. $\int x^r dx = \frac{x^{r+1}}{r+1} + C, \quad \text{provided } r \neq -1$

(To integrate a power of x other than -1, increase the power by 1 and divide by the increased power.)

4.1 Antidifferentiation

THEOREM: Basic Integration Formulas (continued)

3. $\int x^{-1} dx = \int \frac{1}{x} dx = \int \frac{dx}{x} = \ln|x| + C$

4. $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

4.1 Antidifferentiation

Example: Use the Power Rule of Antidifferentiation to determine these indefinite integrals:

a) $\int x^7 dx$; b) $\int x^{99} dx$; c) $\int \sqrt{x} dx$; d) $\int \frac{1}{x^3} dx$

a.)
$$\int x^7 dx = \frac{x^{7+1}}{7+1} + C = \frac{1}{8}x^8 + C$$

Check: $\frac{d}{dx}\left(\frac{1}{8}x^8 + C\right) = \frac{1}{8}\ 8x^7 \ = x^7$

b.)
$$\int x^{99} dx = \frac{x^{99+1}}{99+1} + C = \frac{1}{100}x^{100} + C$$

Check: $\frac{d}{dx}\left(\frac{1}{100}x^{100} + C\right) = \frac{1}{100}\ 100x^{99} \ = x^{99}$

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Example 2 (Continued)

c) We note that $\sqrt{x} = x^{1/2}$. Therefore

$$\int \sqrt{x} dx = \int x^{1/2} = \frac{x^{1/2+1}}{\frac{1}{2}+1} + C = \frac{2}{3}x^{3/2} + C$$

Check: $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + C\right) = \frac{2}{3}\left(\frac{3}{2}x^{1/2}\right) = x^{1/2} = \sqrt{x}$

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Example 2 (Concluded)

d) We note that $\frac{1}{x^3} = x^{-3}$. Therefore

$$\int \frac{1}{x^3}dx = \int x^{-3}dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C$$

Check: $\frac{d}{dx}\left(-\frac{1}{2}x^{-2} + C\right) = -\frac{1}{2}\ \ -2x^{-3}\ \ = x^{-3} = \frac{1}{x^3}$

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Example: Determine the indefinite integral $\int e^{4x}dx.$

Since we know that $\frac{d}{dx}e^x = e^x$, it is reasonable to make this initial guess:

$$\int e^{4x}dx = e^{4x} + C.$$

But this is (slightly) wrong, since

$$\frac{d}{dx}\ \ e^{4x} + C\ \ = 4e^{4x}$$

4.1 Antidifferentiation

Example (Concluded): We modify our guess by inserting $\frac{1}{4}$ to obtain the correct antiderivative:

$$\int e^{4x}dx = \frac{1}{4}e^{4x} + C$$

This checks:

$$\frac{d}{dx}\left(\frac{1}{4}e^{4x} + C\right) = \frac{1}{4}\ \ 4e^{4x}\ \ = e^{4x}$$

4.1 Antidifferentiation

Find each antiderivative:

a.) $\int e^{-3x}dx = \frac{1}{-3}e^{-3x} + C$

b.) $\int e^{1/2\ x}dx = 2e^{1/2\ x} + C$

4.1 Antidifferentiation

THEOREM

Properties of Antidifferentiation

P1. $\int kf(x)dx = k\int f(x)dx$

(The integral of a constant times a function is the constant times the integral of the function.)

P2. $\int\ f(x) \pm g(x)\ \ dx = \int f(x)dx \pm \int g(x)dx$

(The integral of a sum or difference is the sum or difference of the integrals.)

4.1 Antidifferentiation

Example: Determine these indefinite integrals. Assume $x > 0$.

a) $\int\ 3x^5 + 7x^2 + 8\ \ dx$; b) $\int \frac{4 + 3x + 2x^4}{x} dx$

a.) We antidifferentiate each term separately:

$$\int\ 3x^5 + 7x^2 + 8\ \ dx = \int 3x^5 dx + \int 7x^2 dx + \int 8dx$$

$$= 3\left(\frac{1}{6}x^6\right) + 7\left(\frac{1}{3}x^3\right) + 8x$$

$$= \frac{1}{2}x^6 + \frac{7}{3}x^3 + 8x + C$$

4.1 Antidifferentiation

Example (Concluded):

b) We algebraically simplify the integrand by noting that x is a common denominator and then reducing each ratio as much as possible:

$$\frac{4 + 3x + 2x^4}{x} = \frac{4}{x} + \frac{3x}{x} + \frac{2x^4}{x} = \frac{4}{x} + 3 + 2x^3$$

Therefore,

$$\int \frac{4 + 3x + 2x^4}{x} dx = \int\left(\frac{4}{x} + 3 + 2x^3\right)dx$$

$$= 4\ln x + 3x + \frac{1}{2}x^4 + C$$

4.1 Antidifferentiation

Determine these indefinite integrals:

a.) $\int\ 2x^4 + 3x^3 - 7x^2 + x - 5\ \ dx = \frac{2}{5}x^5 + \frac{3}{4}x^4 - \frac{7}{3}x^3 + \frac{1}{2}x^2 - 5x + C$

b.) $\int\ x - 5\ ^2\ dx = \int x^2 - 10x + 25dx = \frac{1}{3}x^3 - 5x^2 + 25x + C$

c.) $\int \frac{x^2 - 7x + 2}{x^2}\ dx = \int 1 - \frac{7}{x} + \frac{2}{x^2}\ dx = x - 7\ln x - \frac{2}{x} + C$

4.1 Antidifferentiation

Example: Find the function f such that

$$f'(x) = x^2 \text{ and } f(-1) = 2.$$

First find $f(x)$ by integrating.

$$f(x) = \int x^2\,dx$$

$$f(x) = \frac{x^3}{3} + C$$

4.1 Antidifferentiation

Example (concluded):

Then, the initial condition allows us to find C.

$$f(-1) = \frac{(-1)^3}{3} + C = 2$$

$$-\frac{1}{3} + C = 2$$

$$C = \frac{7}{3}$$

Thus, $f(x) = \frac{x^3}{3} + \frac{7}{3}$.