Antiderivatives as Areas

4.2

OBJECTIVE

- Find the area under a graph to solve real-world problems
- Use rectangles to approximate the area under a graph.

PEARSON

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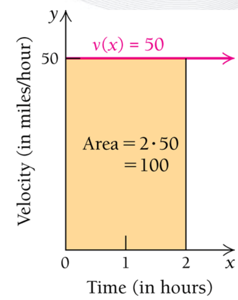
Example 1: A vehicle travels at 50 mi/hr for 2 hr. How far has the vehicle traveled?

The answer is 100 mi. We treat the vehicle's velocity as a function, $v(x) = 50$. We graph this function, sketch a vertical line at $x = 2$, and obtain a rectangle. This rectangle measures 2 units horizontally and 50 units vertically. Its area is the distance the vehicle has traveled:

$$2 \text{ hr} \cdot \frac{50 \text{ mi}}{1 \text{ hr}} = 100 \text{ mi}.$$



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Example 2: The velocity of a moving object is given by the function $v(x) = 3x$, where x is in hours and v is in miles per hour. Use geometry to find the area under the graph, which is the distance the object has traveled:

a.) during the first 3 hr $(0 \le x \le 3)$;

b.) between the third hour and the fifth hour $(3 \le x \le 5)$.

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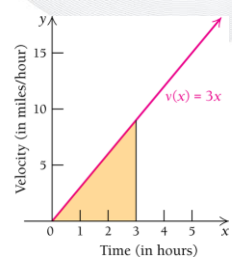
Example 2 (continued):

a.) The graph of the velocity function is shown at the right. We see the region corresponding to the time interval $0 \le x \le 3$ is a triangle with base 3 and height 8 (since $v(3) = 9$). Therefore, the are of this region is

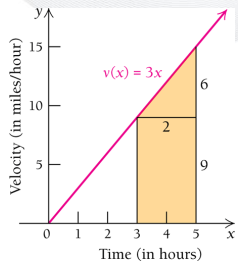


$$A = \frac{1}{2}(3)(9) = \frac{27}{2} = 13.5.$$

The object traveled 13.5 mi during the first 3 hr.

Example 2 (Continued):

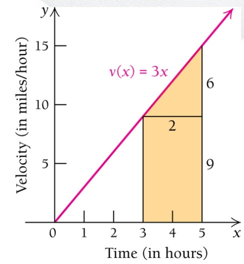
b.) The region corresponding to the time interval $3 \leq x \leq 5$ is a trapezoid. It can be decomposed into a rectangle and a triangle as indicated in the figure to the right. The rectangle has abase 2 and height 9, and thus an area $A = \;\, 2 \quad 9 \;\, = 18.$



Example 2 (Concluded):

b.) The triangle has base 2 and height 6, for an area $A = \frac{1}{2} \;\, 2 \quad 6 \;\, = 6.$

Summing the two areas, we get 24. Therefore, the object traveled 24 mi between the third hour and the fifth hour.

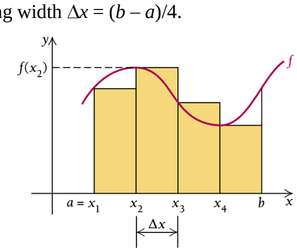


Riemann Sums:

The last two examples, the area function is an antiderivative of the function that generated the graph. Is this always true? Is the formula for the area under the graph of any function that function's antiderivative? How do we handled curved graphs for which area formulas may not be known? We investigate the questions using geometry, in a procedure called *Riemann summation*.

Riemann Sums (continued):

In the following figure, $[a, b]$ is divided into four subintervals, each having width $\Delta x = (b - a)/4$.



The heights of the rectangles are $f(x_1),\ f(x_2),\ f(x_3)$ and $f(x_4)$.

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Riemann Sums (concluded):

The area of the region under the curve is approximately the sum of the areas of the four rectangles:

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

We can denote this sum with summation, or sigma, notation, which uses the Greek capital letter sigma, or Σ:

$$\sum_{i=1}^{4} f(x_i)\Delta x, \text{ or } \sum\nolimits_{i=1}^{4} f(x_i)\Delta x.$$

This is read "the sum of the product $f\;\;x_i\;\;\Delta x$ from $i=1$ to $i=4$." To recover the original expression, we substitute the numbers 1 through 4 successively for i in $f\;\;x_i\;\;\Delta x$ and write plus signs between the results.

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Example 3: Write summation notation for

2 + 4 + 6 + 8 + 10.

Note that we are adding consecutive values of 2.

$$2+4+6+8+10 = \sum_{i=1}^{5} 2i$$

4.2 Antiderivatives as Areas

Quick Check 2

Write the summation notation for each expression

a.) $5+10+15+20+25$ Note that we are adding consecutive multiples of 5. Thus,

$$5+10+15+20+25 = \sum_{i=1}^{5} 5i$$

b.) $33+44+55+66$ Note that we are adding consecutive multiples of 11. Thus,

$$33+44+55+66 = \sum_{i=3}^{6} 11i$$

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Example 4: Write summation notation for:

$$g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_{19})\Delta x.$$

$$g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_{19})\Delta x = \sum_{i=1}^{19} g(x_i)\Delta x$$

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Example 5: Express $\sum_{i=1}^{4} 3^i$ without using summation notation.

$$\sum_{i=1}^{4} 3^i = 3^1 + 3^2 + 3^3 + 3^4 = 120$$

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Quick Check 3

Express $\sum_{i=1}^{6} i^2 + i$ without using summation notation.

$$\sum_{i=1}^{6} i^2 + i = 1^2 + 1 + 2^2 + 2 + 3^2 + 3 + 4^2 + 4 + 5^2 + 5 + 6^2 + 6$$

$$= 2 + 6 + 12 + 20 + 30 + 42$$

$$= 112$$

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Example 6: Express $\sum_{i=1}^{30} h(x_i)\Delta x$ without using summation notation.

$$\sum_{i=1}^{30} h(x_i)\Delta x = h(x_1)\Delta x + h(x_2)\Delta x + \cdots + h(x_{30})\Delta x$$

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Example 7: Consider the graph of:

$$f(x) = 600x - x^2$$

over the interval [0, 600].

a) Approximate the area by dividing the interval into 6 subintervals.

b) Approximate the area by dividing the interval into 12 subintervals.

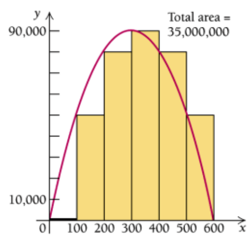
4.2 Antiderivatives as Areas

Example 7 (continued):

a) We divide [0, 600] into 6 intervals of size

$$\Delta x = \frac{600-0}{6} = 100,$$

with x_i ranging from $x_1 = 0$ to $x_6 = 500$.



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Example 7 (continued):

Thus, the area under the curve is approximately

$$\sum_{i=1}^{6} f(x_i)\Delta x = f(0)\cdot 100 + f(100)\cdot 100 + f(200)\cdot 100$$
$$+ f(300)\cdot 100 + f(400)\cdot 100 + f(500)\cdot 100$$
$$= 0\cdot 100 + 50,000\cdot 100 + 80,000\cdot 100$$
$$+90,000\cdot 100 + 80,000\cdot 100 + 50,000\cdot 100$$
$$= 35,000,000$$

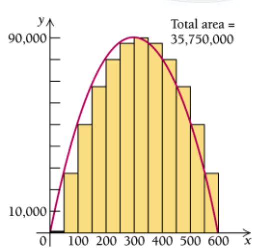
4.2 Antiderivatives as Areas

Example 7 (continued):

b) We divide [0, 600] into 12 intervals of size

$$\Delta x = \frac{600-0}{12} = 50,$$

with x_i ranging from $x_1 = 0$ to $x_{12} = 550$.



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Example 7 (concluded):

Thus, the area under the curve is approximately

$$\sum_{i=1}^{12} f(x_i)\Delta x = f(0)\cdot 50 + f(50)\cdot 50 + f(100)\cdot 50 + f(150)\cdot 50$$
$$+ f(200)\cdot 50 + f(250)\cdot 50 + f(300)\cdot 50 + f(350)\cdot 50$$
$$+ f(400)\cdot 50 + f(450)\cdot 50 + f(500)\cdot 50 + f(550)\cdot 50$$
$$= 0\cdot 50 + 27,500\cdot 50 + 50,000\cdot 50 + 67,500\cdot 50$$
$$+80,000\cdot 50 + 87,500\cdot 50 + 90,000\cdot 50 + 87,500\cdot 50$$
$$+80,000\cdot 50 + 67,500\cdot 50 + 50,000\cdot 50 + 27,500\cdot 50$$
$$= 35,750,000$$

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Section Summary

- The area under a curve can often be interpreted in a meaningful way.
- The units of the area are found by multiplying the units of the input variable by the units of the output variable. It is crucial that the units are consistent.
- Geometry can be used to find areas of regions formed by graphs of linear functions.
- A *Riemann sum* uses rectangles to approximate the area under a curve. The more rectangles, the better approximation.
- The *definite integral* $\int_a^b f(x)\,dx$, is a representation of the exact area under the graph of a continuous function $y = f(x)$, where $f(x) \geq 0$, over an interval $[a, b]$.