Area and Definite Integrals

4.3

OBJECTIVE

- Find the area under a curve over a given closed interval.
- Evaluate a definite integral.
- Interpret an area below the horizontal axis.
- Solve applied problems involving definite integrals.

4.3 Area and Definite Integrals

To find the area under the graph of a nonnegative, continuous function f over the interval $[a, b]$:

1. Find any antiderivative $F(x)$ of $f(x)$. (The simplest is the one for which the constant of integration is 0.)
2. Evaluate $F(x)$ using b and a, and compute $F(b) – F(a)$. The result is the area under the graph over the interval $[a, b]$.

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Example: Find the area under the graph of $y = x^2 + 1$ over the interval $[-1, 2]$.

1. Find any antiderivative $F(x)$ of $f(x)$. We choose the simplest one.

$$F(x) = \frac{x^3}{3} + x$$

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Example (concluded):

2. Substitute 2 and –1, and find the difference $F(2) – F(-1)$.

$$\begin{aligned} F(2)-F(-1) &= \left[\frac{2^3}{3}+2\right]-\left[\frac{(-1)^3}{3}+(-1)\right] \\ &= \frac{8}{3}+2-\left[-\frac{1}{3}-1\right] \\ &= \frac{8}{3}+2+\frac{1}{3}+1 \\ &= 6 \end{aligned}$$

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DEFINITION:

Let f be any continuous function over the interval $[a, b]$ and F be any antiderivative of f. Then, the **definite integral** of f from a to b is

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

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Example: Evaluate $\int_a^b x^2 dx.$

Using the antiderivative $F(x) = x^3/3$, we have

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

It is convenient to use an intermediate notation:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

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Example: Evaluate each of the following:

a) $\int_{-1}^{4} (x^2 - x)\,dx;$

b) $\int_0^3 e^x\,dx;$

c) $\int_1^e \left(1 + 2x - \frac{1}{x}\right) dx \quad (\text{assume } x > 0).$

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Example (continued):

a) $$\int_{-1}^{4} (x^2 - x)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{4}$$

$$= \left(\frac{4^3}{3} - \frac{4^2}{2}\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right)$$

$$= \frac{64}{3} - \frac{16}{2} - \left(-\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{64}{3} - 8 + \frac{1}{3} + \frac{1}{2} = 14\frac{1}{6}$$

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Example (continued):

$$\text{b)} \int_0^3 e^x\,dx = \left[e^x\right]_0^3 = e^3 - e^0 = e^3 - 1$$

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Example (concluded):

$$\text{c)} \int_1^e \left(1 + 2x - \frac{1}{x}\right) dx = \left[x + x^2 - \ln x\right]_1^e$$
$$= (e + e^2 - \ln e) - (1 + 1^2 - \ln 1)$$
$$= (e + e^2 - 1) - (1 + 1 - 0)$$
$$= e + e^2 - 1 - 1 - 1$$
$$= e + e^2 - 3$$

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THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If a continuous function f has an antiderivative F over $[a, b]$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx = F(b) - F(a).$$

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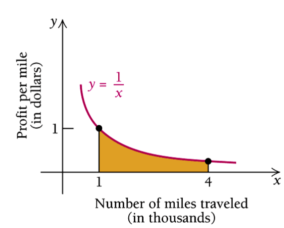
Example: Suppose that y is the profit per mile traveled and x is number of miles traveled, in thousands. Find the area under $y = 1/x$ over the interval [1, 4] and interpret the significance of this area.

$$\int_1^4 \frac{dx}{x} = \ln x \Big|_1^4$$
$$= \ln 4 - \ln 1$$
$$= \ln 4 - 0$$
$$\approx 1.3863$$



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Example (concluded):

The area represents a total profit of $1386.30 when the miles traveled increase from 1000 to 4000 miles.

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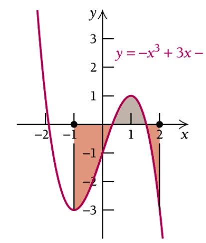
Example: Consider $\int_{-1}^{2}(-x^3+3x-1)dx$. Predict the sign of the integral by examining the graph, and then evaluate the integral.

From the graph, it appears that there is considerably more area below the *x*-axis than above. Thus, we expect that the sign of the integral will be negative.



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Example (concluded):

Evaluating the integral, we have

$$\int_{-1}^{2}(-x^3+3x-1)\,dx=\left[-\frac{x^4}{4}+\frac{3}{2}x^2-x\right]_{-1}^{2}$$

$$=\left(-\frac{2^4}{4}+\frac{3}{2}\cdot 2^2-2\right)-\left(-\frac{(-1)^4}{4}+\frac{3}{2}\cdot(-1)^2-(-1)\right)$$

$$=(-4+6-2)-\left(-\frac{1}{4}+\frac{3}{2}+1\right)$$

$$=0-2\frac{1}{4}=-2\frac{1}{4}$$

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Example: Northeast Airlines determines that the marginal profit resulting from the sale of x seats on a jet traveling from Atlanta to Kansas City, in hundreds of dollars, is given by

$$P'(x)=\sqrt{x}-6.$$

Find the total profit when 60 seats are sold.

Example (continued):
We integrate to find $P(60)$.

$$\begin{aligned} P\ \ 60 &= \int_0^{60} P'\ \ x\ \ dx \\ &= \int_0^{60} \sqrt{x} - 6\ \ dx \\ &= \left[\frac{2}{3}x^{3/2} - 6x\right]_0^{60} \\ &= \left(\frac{2}{3}\cdot 60^{3/2} - 6\cdot 60\right) - \left(\frac{2}{3}\cdot 0^{3/2} - 6\cdot 0\right) \\ &\approx -50.1613 \end{aligned}$$

Example (concluded):
When 60 seats are sold, Northeast's profit is –\$5016.13. That is, the airline will lose \$5016.13 on the flight.

Example: A particle starts out from some origin. Its velocity, in miles per hour, is given by

$$v(t) = \sqrt{t} + t,$$

where t is the number of hours since the particle left the origin. How far does the particle travel during the second, third, and fourth hours (from $t = 1$ to $t = 4$)?

Example (continued):
Recall that velocity, or speed, is the rate of change of distance with respect to time. In other words, velocity is the derivative of the distance function, and the distance function is an antiderivative of the velocity function. To find the total distance traveled from $t = 1$ to $t = 4$, we evaluate the integral

$$\int_1^4 \sqrt{t} + t\ \ \ dt.$$

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Example (concluded):

$$\int_1^4 \sqrt{t}+t \ dt = \int_1^4 t^{1/2}+t \ dt$$

$$= \left[\frac{2}{3}t^{3/2}+\frac{1}{2}t^2\right]_1^4$$

$$= \frac{2}{3}\cdot 4^{3/2}+\frac{1}{2}\cdot 4^2-\left(\frac{2}{3}\cdot 1^{3/2}+\frac{1}{2}\cdot 1^2\right)$$

$$= \frac{16}{3}+\frac{16}{2}-\frac{2}{3}-\frac{1}{2}=\frac{14}{3}+\frac{15}{2}=\frac{73}{6}$$

$$= 12\frac{1}{6} \text{ mi.}$$