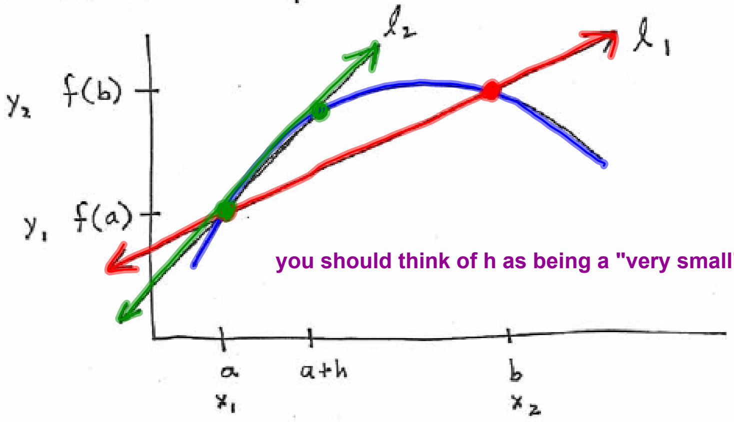
3.1 and 3.2 Derivative (Differentiation)

Definition: A <u>secant line</u> to a curve is a line that intersects the curve at two or more points.



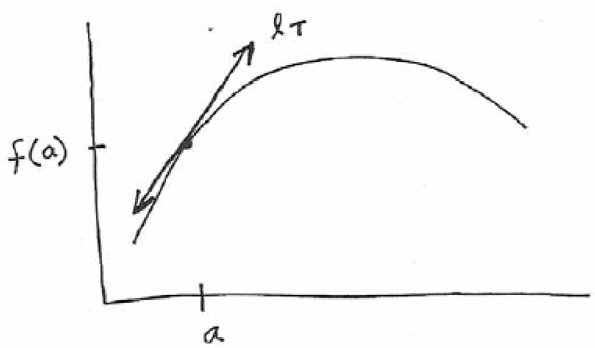
you should think of h as being a "very small" positive number.

the textbook sometimes uses Δx instead of h

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$

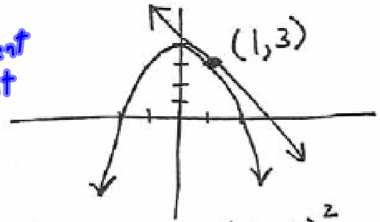
$$m_2 = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

Definition: A <u>tangent line</u> to a curve is a line that best approximates the curve at a single point.
But two points determine a line!!!



$$m_T = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example: Find the equation of the tangent line to the function $f(x) = 4 - x^2$ at the point $(1, 3)$.



Different Quotient

let $a = 1$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4 - (1+h)^2 - f(1)}{h}$$

$$= \lim_{h \to 0} \left[\frac{4 - (1 + 2h + h^2) - 3}{h} = \frac{4 - 1 - 2h - h^2 - 3}{h} \right.$$

$$\left. = \frac{3 - 2h - h^2 - 3}{h} = \frac{-2h - h^2}{h} = \frac{\cancel{h}(-2 - h)}{\cancel{h}} = -2 - h \right]$$

$$= -2 \;; \qquad y - y_1 = m(x - x_1) \implies y - 3 = -2(x - 1)$$

$$\implies y = -2x + 5$$

Definition: The generalization of the formula for finding the slope of the tangent line at any given x-coordinate is called the derivative. The derivative of $f(x)$ is often denoted $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Returning to the last problem, find the derivative of $f(x) = 4 - x^2$. Also, find the slope of the tangent line when $x = -2, -1, 0, 1, 2,$ and 5.

$$f'(x) = \lim_{h\to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{difference quotient (DQ)}}$$

If $f(x) = 4 - x^2$ then $f(x+h) = 4 - (x+h)^2$

$$= 4 - (x^2 + 2xh + h^2)$$

$$= 4 - x^2 - 2xh - h^2$$

$$\text{D.Q.} = \frac{f(x+h) - f(x)}{h} = \frac{4 - x^2 - 2xh - h^2 - (4 - x^2)}{h}$$

$$= \frac{\cancel{4} - \cancel{x^2} - 2xh - h^2 - \cancel{4} + \cancel{x^2}}{h} = \frac{-2xh - h^2}{h}$$

$$= \frac{h(-2x - h)}{h} = -2x - h.$$

$$f'(x) = \lim_{h\to 0} \text{D.Q.} = \lim_{h\to 0}(-2x - h) = -2x$$

the slopes are

$f'(-2) = 4$, $f'(-1) = 2$, $f'(0) = 0$

$f(1) = -2$, $f'(2) = -4$, $f'(5) = -10$

Ex. Let $f(x) = \frac{2}{x}$, find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$$

$$= \lim_{h \to 0} \left[\frac{2x - 2(x+h)}{x(x+h)} \div h \right] = \lim_{h \to 0} \left[\frac{2x - 2x - 2h}{x(x+h)} \cdot \frac{1}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-2\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} \right] = \lim_{h \to 0} \frac{-2}{x(x+h)} = \frac{-2}{x^2}$$

Ex. Let $f(x) = x^3$, find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} = 3x^2$$

Ex. Let $f(x) = \sqrt{x}$, find $f'(x)$

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right]$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}}{2x}$$

Pascal's Triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

$(x+h)^6 = ?$

$x^6 + 6x^5h + 15x^4h^2 +$
$20x^3h^3 + 15x^2h^4 +$
$6xh^5 + h^6$

Find $f'(x)$ if $f(x) = 9x^6$

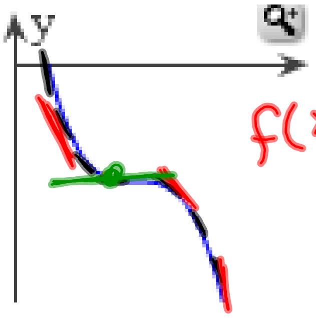
$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \ldots =$

$(x+h)^6 = ?$

$x^6 + 6x^5h + 15x^4h^2 +$
$20x^3h^3 + 15x^2h^4 +$
$6xh^5 + h^6$

$54x^5$

Graph the derivative of the function graphed on the right.



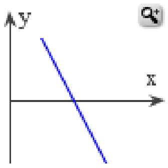
f(x)

all slopes are nonpositive

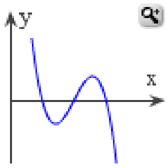
$y = f'(x) \le 0$

Choose the correct graph below. $f'(x)$

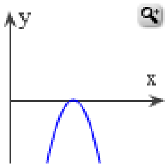
A.



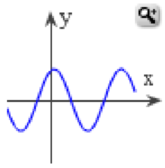
B.



C.



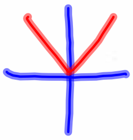
D.



Differentiability

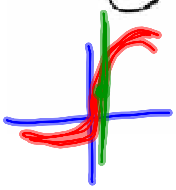
$f(x)$ does not have a derivative at $x=c$ if at least one of the following is true

1. c is a point of discontinuity



2. the sign of the slope of the tangent line as $x \to c^-$ is different from the slope of the tangent line as $x \to c^+$.

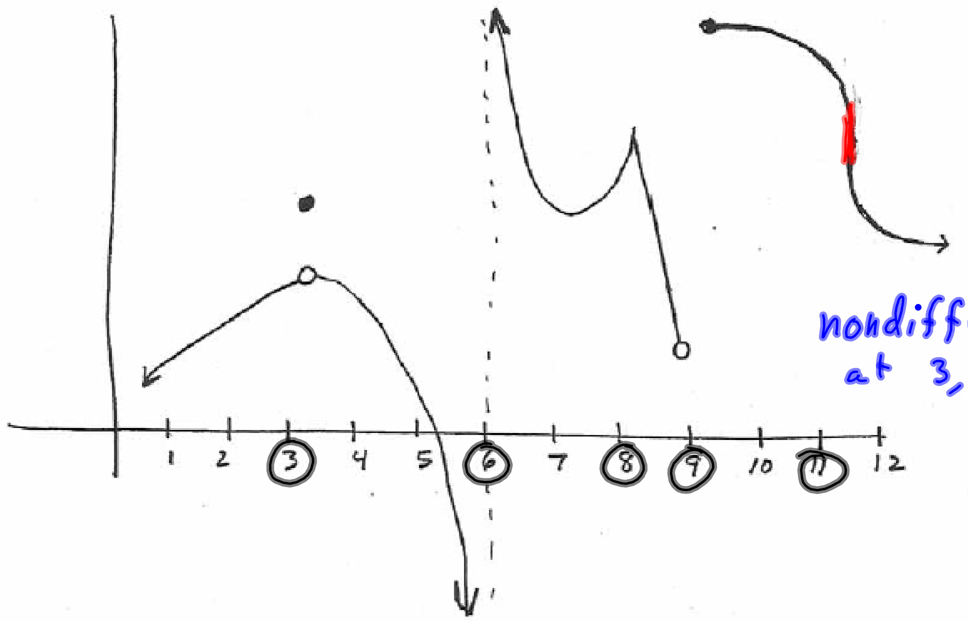
Ex. $\boxed{f(x) = |x|}$ is nondifferentiable at $x = 0$.
(NONDIFFERENTIABLE AT SHARP POINTS)



3. the tangent line to the curve is a vertical line.

Ex. $f(x) = \sqrt[3]{x}$ is nondifferentiable at $x = 0$.

Where is the following function nondifferentiable



nondifferentiable at 3, 6, 8, 9, 11

note: If $f(x)$ is differentiable at c then $f(x)$ is continuous at c.

~~Ap~~ Application

Average Rate of Change $= \dfrac{f(b) - f(a)}{b - a}$ ↖ difference quotient

Say you finished a car race. The car's distance in feet from the finishing line after t seconds is $f(t) = 160t - 5t^2$ where $0 \le t \le 16$.

Thus, $f(0) = 0$, $f(4) = 560$, $f(7) = 875$

$f(7.5) = 918.75$, $f(7.95) = 955.988$

$f(8) = 960$

Avg velocity from $\overline{\text{a to b}}$ 0 to 8: $\dfrac{f(8) - f(0)}{8 - 0} = \dfrac{960 - 0}{8} = \dfrac{120 \text{ ft}}{\text{sec}}$

Avg velocity from 4 to 8: $\dfrac{f(8) - f(4)}{8 - 4} = \dfrac{960 - 560}{4} = \dfrac{100 \text{ ft}}{\text{sec}}$

Avg velocity from 7 to 8: $\dfrac{f(8) - f(7)}{8 - 7} = 960 - 875 = \dfrac{85 \text{ ft}}{\text{se}}$

Avg velocity from 7.5 to 8: $\dfrac{f(8) - f(7.5)}{8 - 7.5} = \dfrac{960 - 918.75}{0.5} = \dfrac{82.5 \text{ ft}}{\text{se}}$

Avg velocity from 7.95 to 8: $\dfrac{f(8) - f(7.95)}{8 - 7.95} = \dfrac{960 - 955.988}{.05} = \dfrac{80.24 \text{ ft}}{\text{sec}}$

What should the speedometer display after 8 seconds?

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Instantaneous Rate of Change =

$$\lim_{(b-a)\to 0} \frac{f(b)-f(a)}{b-a} \qquad \text{let } h = b-a \Rightarrow b = a+h$$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

For our car, the instantaneous velocity at $t = 8$ is

$$\lim_{h\to 0} \frac{f(8+h)-f(8)}{h} \quad ; \text{ recall, } f(t) = 160t - 5t^2$$

$$\begin{aligned} f(8+h) &= 160(8+h) - 5(8+h)^2 \\ &= 1280 + 160h - 5(64 + 16h + h^2) \\ &= 1280 + 160h - 320 - 80h - 5h^2 \\ &= 960 + 80h - 5h^2 \end{aligned}$$

$$f(8) = 960$$

Thus, instantaneous velocity is

$$\lim_{h\to 0}\left[\frac{f(8+h)-f(8)}{h} = \frac{960 + 80h - 5h^2 - 960}{h} = \frac{80h - 5h^2}{h} = \frac{h(80-h)}{h} = 80 - h\right] = 80 \text{ ft/sec}$$

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