5.1 Area Using Finite Sums/Riemann Sums

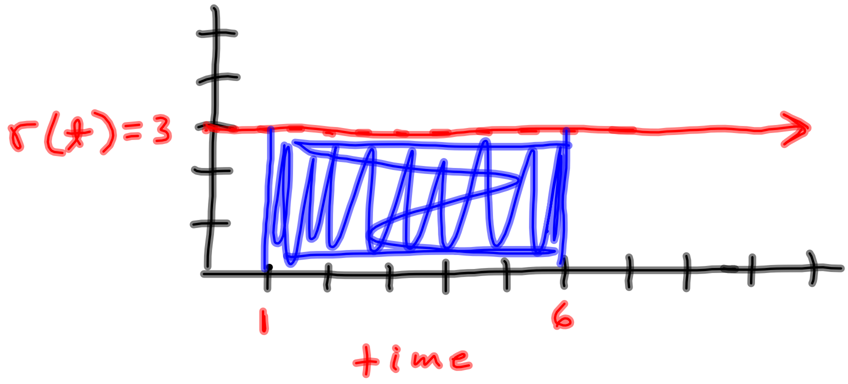
Georg Riemann's Thought

Riemann watched sand drain from a sand bag that had a sizable hole in one of its corners. He estimated that the flow rate was approximately r(t). He challenged himself to determine the amount of sand that drains from the bag during a given time period - for example, between 1 and 6 seconds

$r(t) = \sqrt{10-t}\ , \quad 0 \le t \le 10. \quad (oz/sec)$

If r (t) were constant, say r(t) = 3 oz per sec, then this problem would be EASY.

$V(t) = r\,t$

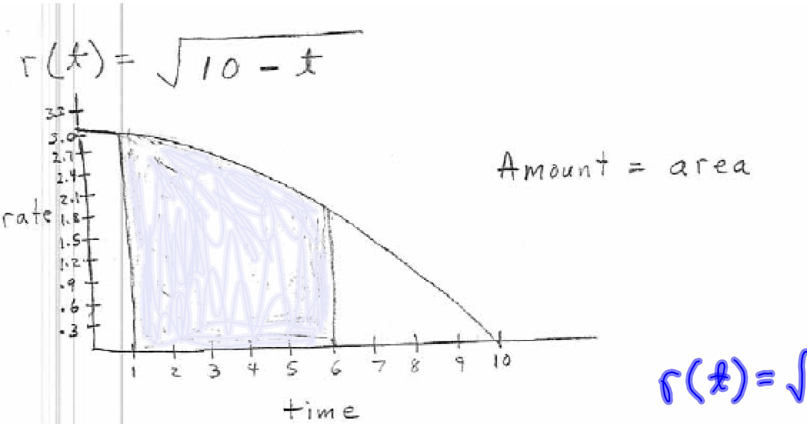


In this EASY scenario, Volume of

sand is the area of blue rectangle.

$V = (3\ oz/sec)(6\,sec - 1\ sec)$
$= \boxed{15\ ounces}$

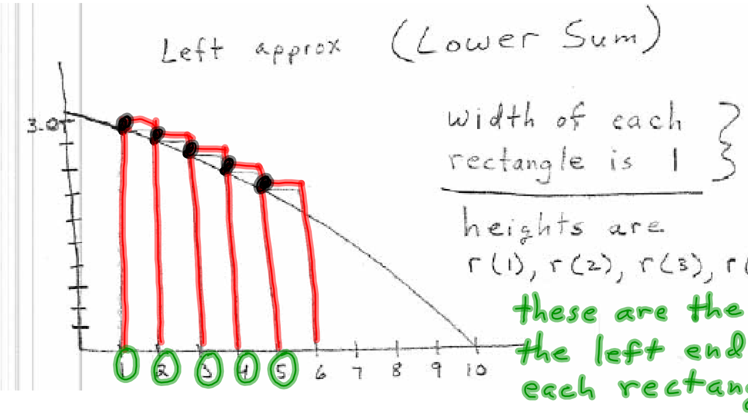
Unfortunately, Riemann's Scenario
does not have a constant rate of flow.



$r(t) = \sqrt{10 - t}$

The answer is the area of the shaded region;
but, how does one find this area?

If $\Delta t = \frac{1}{2}$ then
I would have
10 rectangles



$\boxed{\Delta t = 1}$

these are the heights of
the left endpoints of
each rectangle.

The sum of the areas of each rectange would overestimate

the area under the curve. It is sometimes called the lower

sum because the estimate would be an underestimate if the function

were an increasing function.

$$A = \int_1^6 r(t)\,dt \approx \sum_{i=1}^{6} r(t_i)\,\Delta t$$

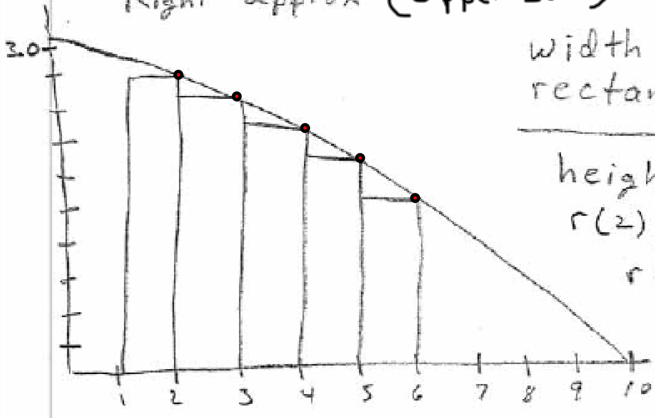
recall, $\Delta t = 1$

$$\approx r(1) + r(2) + r(3) + r(4) + r(5)$$

$$= 3 + \sqrt{8} + \sqrt{7} + \sqrt{6} + \sqrt{5} \approx \boxed{13.16}$$

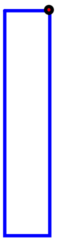
```
NORMAL FLOAT AUTO a+bi DEGREE CL
3+√(8)+√(7)+√(6)+√(5)
                      13.15973616
```

Right approx (Upper Sum)



width of each rectangle is 1 } $\Delta x = 1$

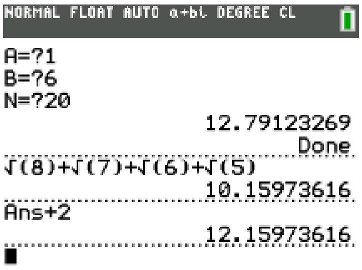
heights are
r(2), r(3), r(4),
r(5), r(6)



$$r(2) + r(3) + r(4) + r(5) + r(6) =$$

$$\sqrt{8} + \sqrt{7} + \sqrt{6} + \sqrt{5} + \sqrt{4}$$

≈ 12.16



```
NORMAL FLOAT AUTO a+bi DEGREE CL
A=?1
B=?6
N=?20
                        12.79123269
                               Done
√(8)+√(7)+√(6)+√(5)
                        10.15973616
Ans+2
                        12.15973616
```

lower (left) : $\Delta x \sum_{i=0}^{n-1} f(x_i)$

upper (right) : $\Delta x \sum_{i=1}^{n} f(x_i)$

Note avg of
13.16 and 12.16
you get 12.66

where $\Delta x = \frac{b-a}{n}$

$a = x_0 < x_1 < x_2 < \dots < x_n = b$

$x_{i+1} - x_i = \Delta x$

Trapezoid Method

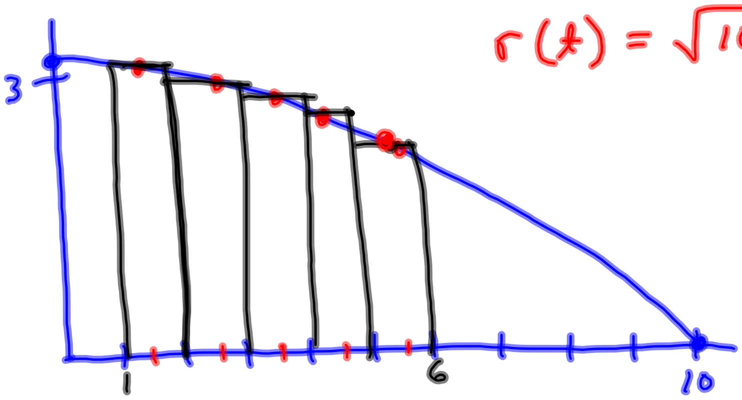
$$\text{Avg} = \frac{13.16 + 12.16}{2} = 12.66$$

Middle Sum Technique

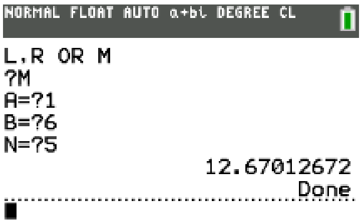
$r(t) = \sqrt{10 - t}$

$n = 5$

$\Delta t = 1$

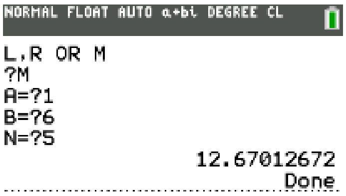


$\sqrt{8.5} + \sqrt{7.5} + \sqrt{6.5} + \sqrt{5.5} + \sqrt{4.5}$

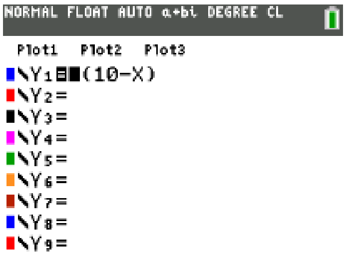


```
NORMAL FLOAT AUTO a+bi DEGREE CL
L,R OR M
?M
A=?1
B=?6
N=?5
                    12.67012672
                           Done
```

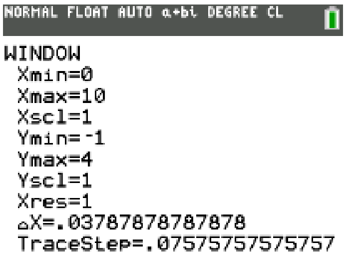
Calculator Programs and built in calculator program (Simpson's Rule).



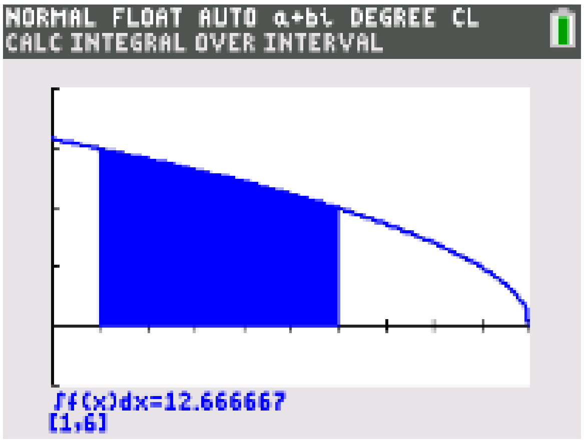
```
NORMAL FLOAT AUTO a+bi DEGREE CL
L,R OR M
?M
A=?1
B=?6
N=?5
                12.67012672
                       Done
```



```
NORMAL FLOAT AUTO a+bi DEGREE CL
 Plot1  Plot2  Plot3
■\Y1=■(10-X)
■\Y2=
■\Y3=
■\Y4=
■\Y5=
■\Y6=
■\Y7=
■\Y8=
■\Y9=
```



```
NORMAL FLOAT AUTO a+bi DEGREE CL
WINDOW
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=-1
 Ymax=4
 Yscl=1
 Xres=1
 ΔX=.03787878787878
 TraceStep=.07575757575757
```



```
NORMAL FLOAT AUTO a+bi DEGREE CL
CALC INTEGRAL OVER INTERVAL
∫f(x)dx=12.666667
[1,6]
```

In a subsequent section, we will find that the EXACT area underneath a nonnegative continuous function on (a, b) is F(b)- F-(a) where F'(x) = f(x). F(x) is called an antiderivative.(Section 4.7)

For our problem $f(x) = \sqrt{10-x}$ on $[1,6]$,
Is $F(x) = \frac{-2}{3}(10-x)^{3/2}$ an antiderivative?

$$F'(x) = \frac{3}{2} \cdot \frac{-2}{3}(10-x)^{3/2-1}(-1)$$
$$= (-1)(10-x)^{\frac{1}{2}}(-1) = \sqrt{10-x}.$$

Yes.

Thus, Area = F(6) − F(1).

$$F(6) = \frac{-2}{3}(10-6)^{3/2} = \frac{-2}{3}\sqrt{4^3} = \frac{-2}{3}(8) = \frac{-16}{3}$$
$$F(1) = \frac{-2}{3}(10-1)^{3/2} = \frac{-2}{3}\sqrt{9^3} = \frac{-2}{3}(27) = -18$$
$$F(6) - F(1) = \frac{-16}{3} + \frac{54}{3} = \frac{38}{3} = \boxed{12\frac{2}{3}}$$

Theorem: If $f(x) \geq 0$ is continuous on $[a,b]$ then the area under the curve is $F(b)-F(a)$ where $F'(x) = f(x)$.

Quasi Proof:

We have seen that area $\overset{(=,\, n\to\infty)}{\approx} \Delta x \sum_{i=0}^{n-1} f(x_i)$ lower sum

where $\Delta x = \dfrac{b-a}{n}$, $a = x_0 < x_1 < \ldots < x_n = b$. $\Delta x \to 0$

If $F'(x) = f(x)$ then

$$f(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \qquad x_1 = x_0 + \Delta x$$

$$\text{area} = \Delta x \sum_{i=0}^{n-1} \lim_{\Delta x \to 0} \frac{F(x_i + \Delta x) - F(x_i)}{\Delta x}$$

$$= \sum_{i=0}^{n-1} \left[F(x_i + \Delta x) - F(x_i) \right] \text{ as } \quad n \to \infty \Rightarrow \boxed{\Delta x \to 0}$$

$$= F(\overset{F(x_1)}{x_0 + \Delta x}) - F(\overset{F(a)}{x_0}) + F(\overset{F(x_2)}{x_1 + \Delta x}) - F(x_1)$$

$$F(x_2 + \Delta x) - F(x_2) + F(x_3 + \Delta x) - F(x_3)$$

$$\vdots$$

$$F(\overset{F(x_3)}{x_{n-2} + \Delta x}) - F(x_{n-2}) + F(\overset{F(x_4)}{x_{n-1} + \Delta x}) - F(x_{n-1}$$

$$= F(x_1) - \overset{F(x_{n-1})}{F(a)} + F(x_2) - \overset{F(x_n) = F(b)}{F(x_1)} +$$

$$F(x_3) - F(x_2) + F(x_4) - F(x_3) +$$

$$\vdots$$

$$F(x_{n-1}) - F(x_{n-2}) + F(b) - F(x_{n-1})$$

$$= F(b) - F(a).$$

Q.E.D.

Jun 18-8:57 AM

Ex. Estimate area under $f(x) = \sin x$ on $[\overset{a}{0}, \overset{b}{\frac{\pi}{2}}]$ using 4 rectangles using lower & upper sums.

$$\Delta x = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$$

$x_0 = 0$, $x_1 = \frac{\pi}{8}$, $x_2 = \frac{\pi}{4}$, $x_3 = \frac{3\pi}{8}$, $x_4 = \frac{\pi}{2}$

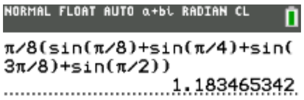
$$L: A \approx \Delta x \sum_{i=0}^{3} f(x_i) = \frac{\pi}{8}\left(\sin 0 + \sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8}\right)$$

$$\approx \boxed{0.79}$$

```
NORMAL FLOAT AUTO a+bi RADIAN CL
π/8(sin(0)+sin(π/8)+sin(π/
4)+sin(3π/8))
                  .7907662601
```

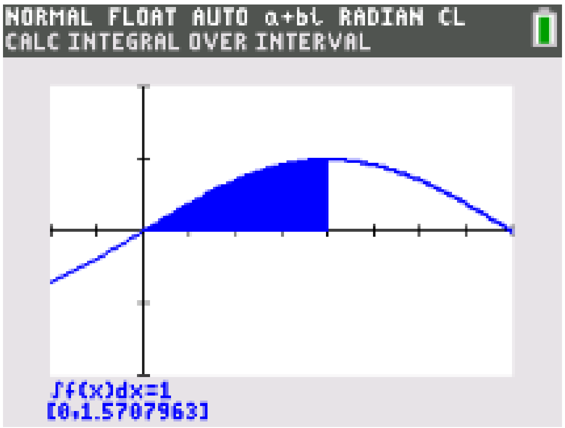
$$R: A \approx \Delta x \sum_{i=1}^{4} f(x_i) = \frac{\pi}{8}\left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2}\right)$$

$$\approx \boxed{1.18}$$



```
NORMAL FLOAT AUTO a+bi RADIAN CL
π/8(sin(π/8)+sin(π/4)+sin(
3π/8)+sin(π/2))
                  1.183465342
```

$$\text{Trapezoid} = \frac{0.79 + 1.18}{2} \approx \boxed{.99}$$



```
NORMAL FLOAT AUTO a+bi RADIAN CL
CALC INTEGRAL OVER INTERVAL
∫f(x)dx=1
[0,1.5707963]
```

note: $f(x) = \sin x$

$F(x) = -\cos x$

$$A = F\left(\frac{\pi}{2}\right) - F(0)$$

$$= -\cos\left(\frac{\pi}{2}\right) - \left(-\cos(0)\right)$$

$$= 0 - (-1) = \boxed{1}$$