7.2 Natural Logarithms ($\ln x = \log_e x$)

Recall, $y = \ln x \iff x = e^y$

where $e \approx 2.718281828459\ldots = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

{euler ~~constant~~ number pronounced oiler

$= \lim_{h\to 0}(1+h)^{\frac{1}{h}}$

Thus, $\ln 100 = 4.605170186$

means $2.718281828459^{4.605170186} = 100$

Also recall,

1. $\log_b MN = \log_b M + \log_b N$

   $\ln MN = \ln M + \ln N$

   Ex. $\ln[(2x+3)(7x-1)] =$

   $\ln(2x+3) + \ln(7x-1)$

2. $\log_b \frac{M}{N} = \log_b M - \log_b N$

   $\ln \frac{M}{N} = \ln M - \ln N$

   Ex. $\ln \frac{x(x-2)}{(x+3)} =$

   $\ln x + \ln(x-2) - \ln(x+3)$

3. $\log_b M^p = p\log_b M$

   $\ln M^p = p\ln M$

   Ex. $\ln \frac{x^2(x+5)^3}{\sqrt{2x-3}} =$

   $\ln x^2 + \ln(x+5)^3 - \ln(2x-3)^{\frac{1}{2}} =$

   $2\ln x + 3\ln(x+5) - \frac{1}{2}\ln(2x-3)$

4. $\log_b x = \frac{\ln x}{\ln b}$

   Ex. $\log_2 1024 = \frac{\ln 1024}{\ln 2} = 10$

   which means $2^{10} = 1024$.

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$$\frac{d}{dx}\ln x = ?$$

$$\frac{d}{dx}(\ln x) = \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[\frac{\ln\left(\frac{x+\Delta x}{x}\right)}{\Delta x} = \frac{1}{\Delta x}\ln\left(1+\frac{\Delta x}{x}\right)\right]$$

$$= \lim_{\Delta x \to 0}\left(\ln\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}\right)$$

$*$ let $\Delta x = \frac{x}{n}$; $n \to \infty$ $\quad \frac{n}{x} = \frac{1}{\Delta x}$

$$= \lim_{n \to \infty}\left[\ln\left(1+\frac{1}{n}\right)^{\frac{n}{x}} = \ln\left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{1}{x}}\right]$$

$$= \lim_{n \to \infty}\left[\frac{1}{x}\ln\left(1+\frac{1}{n}\right)^n\right] = \lim_{n \to \infty}\frac{1}{x}\ln e$$

$$= \lim_{n \to \infty}\frac{1}{x} = \frac{1}{x} \qquad \text{QED}$$

Differentiation Rule for $y = \ln x$

① $\frac{d}{dx}(\ln x) = \frac{1}{x}$

② $\frac{d}{dx}(\ln u) = \frac{u'}{u}$

Ex. Find $f'(x)$ if

$f(x) = \ln(x^2 + 2x - 1)$

$$f'(x) = \frac{2x+2}{x^2+2x-1} = \frac{2(x+1)}{x^2+2x-1}$$

$f(x) = \ln(\sin x)$

$$f'(x) = \frac{\cos x}{\sin x} = \cot x$$

Find $f'(x)$ if

$$f(x) = \frac{\ln x}{x}; \quad f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f(x) = x \cdot (\ln x)^3; \quad f'(x) = x\left[3(\ln x)^2 \cdot \frac{1}{x}\right] + (\ln x)^3 \cdot 1 = 3(\ln x)^2 + (\ln x)^3 = (\ln x)^2(3 + \ln x)$$

GCF

$$f(x) = \ln\left(\frac{x^2}{3x-1}\right)$$

$$= \ln x^2 - \ln(3x-1)$$

$$= 2\ln x - \ln(3x-1)$$

$$f'(x) = 2 \cdot \frac{1}{x} - \frac{3}{3x-1} = \frac{2}{x} - \frac{3}{3x-1}$$

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$$f(x) = \ln\left[\frac{\sqrt{x}}{(x+1)^3(x-1)^4}\right]$$

Rewrite first

$$= \ln x^{\frac{1}{2}} - \ln(x+1)^3 - \ln(x-1)^4$$

$$= \frac{1}{2}\ln x - 3\ln(x+1) - 4\ln(x-1)$$

$$f'(x) = \frac{1}{2}\cdot\frac{1}{x} - 3\cdot\frac{1}{x+1} - 4\cdot\frac{1}{x-1}$$

$$= \frac{1}{2x} - \frac{3}{x+1} - \frac{4}{x-1}$$

Silly

$$y = \underline{\underline{4x}}$$

$$\ln y = \ln 4x$$

$$\ln y = \ln 4 + \ln x$$

$$y\left(\frac{1}{y}\cdot\frac{dy}{dx}\right) = y\left(0 + \frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{y}{x} = \frac{4x}{x} = 4$$

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Find $y'(x)$ if $y = \dfrac{\sin x \cos x}{x}$

Method 1: Use Quotient Rule

$$y' = \frac{x\left[\sin x(-\sin x) + \cos x \cos x\right] - \sin x \cos x\,(1)}{x^2}$$

$$= \frac{x\cos^2 x - x\sin^2 x - \sin x \cos x}{x^2}$$

$$= \frac{x(\cos^2 x - \sin^2 x)}{x^2} - \frac{\sin x \cos x}{x^2}$$

$$= \frac{\cos^2 x - \sin^2 x}{x} - \frac{\sin x \cos x}{x^2}$$

$$y = \frac{\sin x \cos x}{x}$$

Method 2:

$$\ln y = \ln\left(\frac{\sin x \cos x}{x}\right)$$

$$\ln y = \ln \sin x + \ln \cos x - \ln x$$

$$\frac{y'}{y} = \frac{\cos x}{\sin x} + \frac{-\sin x}{\cos x} - \frac{1}{x}$$

$$y' = y\left(\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} - \frac{1}{x}\right)$$

$$= \frac{\sin x \cos x}{x}\left(\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} - \frac{1}{x}\right)$$

$$= \frac{\cos^2 x}{x} - \frac{\sin^2 x}{x} - \frac{\sin x \cos x}{x^2}$$

$$y = \frac{(4x-7)^5(2x+1)^3}{\sqrt[3]{5x+6}}$$

$$\ln y = \ln(4x-7)^5 + \ln(2x+1)^3 - \ln(5x+6)^{\frac{1}{3}}$$

$$\ln y = 5\ln(4x-7) + 3\ln(2x+1) - \frac{1}{3}\ln(5x+6)$$

$$\frac{y'}{y} = 5\cdot\frac{4}{4x-7} + 3\cdot\frac{2}{2x+1} - \frac{1}{3}\cdot\frac{5}{5x+6}$$

$$y' = \frac{(4x-7)^5(2x+1)^3}{\sqrt[3]{5x+6}}\left(\frac{20}{4x-7} + \frac{6}{2x+1} - \frac{5}{15x+18}\right)$$

Recall, $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

$$\int x^{-1} \, dx = \int \frac{dx}{x} = \ln|x| + C$$

Ex. Find $\int \frac{dx}{2x+1} =$

let $u = 2x+1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$$\int \frac{1}{u} \cdot \frac{du}{2} =$$

$$\frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln|u| + C$$

or

$$\ln \sqrt{u} + C$$

$$= \ln \sqrt{2x+1} + C$$

Ex. Find $\int_{x=0}^{x=\frac{\pi}{2}} \frac{\cos x}{\sin x + 3} \, dx =$

let $u = \sin x + 3$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

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NORMAL FLOAT AUTO REAL RADIAN CL
ln(4/3)
                  .2876820725
ln(4)-ln(3)
                  .2876820725
fnInt(cos(X)/(sin(X)+3),X,
0,π/2)
                  .2876820725
```

$$\int_{u=3}^{u=4} \frac{\cancel{\cos x}}{u} \cdot \frac{du}{\cancel{\cos x}} =$$

$$\int_3^4 \frac{du}{u} = \ln u \Big|_3^4 = \ln 4 - \ln 3$$

$$= \ln \frac{4}{3}$$

Ex. $\int \frac{\sec^2 x}{5 + 4\tan x} \, dx =$

let $u = 5 + 4\tan x$

$\frac{du}{dx} = 4\sec^2 x$

$dx = \frac{du}{4\sec^2 x}$

$$\frac{1}{4} \int \frac{du}{u} = \frac{1}{4} \ln|u| + C$$

$$= \frac{1}{4} \ln|5 + 4\tan x| + C$$

Find $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$$= -\int \frac{du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

OR

$$= \ln|\sec x| + C$$

Find $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 3 \cot \frac{x}{2} \, dx$

let $u = \sin\left(\frac{1}{2}x\right)$

$\frac{du}{dx} = \frac{1}{2}\cos\left(\frac{1}{2}x\right)$

$dx = \frac{2du}{\cos\left(\frac{1}{2}x\right)}$

$= \frac{2du}{\cos \frac{x}{2}}$

$$3 \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} \, dx$$

$$6 \int_{\frac{1}{2}}^{\sqrt{3}/2} \frac{du}{u} = 6 \ln u \Big|_{\frac{1}{2}}^{\sqrt{3}/2}$$

$$= 6\left[\ln \sqrt{3}/2 - \ln \tfrac{1}{2}\right]$$

$$= 6\left[\ln \sqrt{3}\right] = 3 \ln 3$$

$$= \ln 27$$

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NORMAL FLOAT AUTO REAL RADIAN CL
ln(27)
                        3.295836866
fnInt(3/tan(X/2),X,π/3,2π/
3)
                        3.295836866
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Ex. $\int_{e}^{e^5} \frac{dx}{x \ln x} =$

Find the area between ln x and ln 2x on the interval from x=1 to x=2