7.3 Exponential Functions

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \approx 2.718281828459$$

$$\left(1+\frac{1}{100}\right)^{100} \approx 2.704813829$$

$$\left(1+\frac{1}{\text{million}}\right)^{\text{million}} \approx 2.718280469$$

$$\left(1+\frac{1}{\text{trillion}}\right)^{\text{trillion}} \approx 2.718281827$$

Differentiation/Integration Rules for the Euler number

① $\frac{d}{dx}e^x = e^x$

② $\frac{d}{dx}e^u = u'e^u$

③ $\int e^x\,dx = e^x + C$

④ $\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$

Ex. Find $\frac{dy}{dx}$

$y = e^{\sin 2x} \Rightarrow y' = \frac{d}{dx}\sin 2x \cdot e^{\sin 2x}$

$= 2\cos 2x \cdot e^{\sin 2x}$

$y = e^{x^2} \Rightarrow \frac{dy}{dx} = 2xe^{x^2}$

$y = e^{9x^2+3x-1} \Rightarrow \frac{dy}{dx} = (18x+3)e^{9x^2+3x-1}$

OR

$3(6x+1)e^{9x^2+3x-1}$

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Find $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$

let $u = x^{-1}$

$\frac{du}{dx} = -x^{-2}$

$dx = \frac{du}{-x^{-2}}$

$dx = -x^2\,du$

$$\int \frac{e^u}{\cancel{x^2}} \cdot \left(-\cancel{x^2}\,du\right) =$$

$$-\int e^u\,du = -e^u + c$$

$$= -e^{\frac{1}{x}} + c$$

Find $\displaystyle\int_0^{\ln 8} \frac{e^x}{1+e^x}\,dx$

let $u = 1 + e^x$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$$\int_2^9 \frac{du}{u} = \ln u \Big|_2^9$$

$$= \ln 9 - \ln 2$$

$$= \ln \frac{9}{2} = \ln(4.5)$$

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NORMAL FLOAT AUTO REAL RADIAN CL
ln(4.5)
                    1.504077397
fnInt(e^(X)/(1+e^(X)),X,0,
ln(8))
                    1.504077397
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On "old" calculators and programming languages you could NOT use 2^3 for 2^3. The input for this was exp(3 · ln(2)) $= e^{3\ln 2}$

Theorem: $e^{x\ln a} = a^x$

Proof: $e^{x\ln a} = e^{\ln a^x}$.

Since $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions, $e^{\ln a^x} = a^x$.

Find $\frac{dy}{dx}$ if $y = 2^x$.

$y = e^{x\ln 2} \Rightarrow \frac{dy}{dx} = (\ln 2)e^{x\ln 2}$
$= (\ln 2)\cdot 2^x$.

More Rules: $\frac{d}{dx}(a^x) = (\ln a)a^x$.

$$\frac{d}{dx}(a^u) = \underline{\underline{u'(\ln a)a^u}}$$

Ex. Find $\frac{dy}{dx}$, if $y = 14^{\sin x}$

$\frac{dy}{dx} = \cos x(\ln 14)14^{\sin x}$

Recall the change of base formula:

$\log_a x = \frac{\ln x}{\ln a}$; thus, $\log_7 38 = \frac{\ln 38}{\ln 7}$.

Ex. If $y = \log_7 x$ find $\frac{dy}{dx}$

$$y = \frac{\ln x}{\ln 7} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 7} \cdot \frac{1}{x}$$

$$= \frac{1}{x \ln 7}$$

Even More Rules

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx} \log_a u = \frac{u'}{u \ln a}$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

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Find $\frac{dy}{dx}$ if

$$y = \log_{18} \sin(4x)$$

$$\frac{dy}{dx} = \frac{4\cos 4x}{(\sin 4x)\cdot \ln 18} \quad \text{OR} \quad \frac{4\cot 4x}{\ln 18}$$

Find $\int \cos x \cdot 2^{\sin x}\, dx$

let $u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$$\int 2^u\, du = \frac{2^u}{\ln 2} + C$$

$$= \frac{2^{\sin x}}{\ln 2} + C$$