7.5 L'Hôpital Rule

The indeterminate forms include 0^0, $0/0$, 1^∞, $\infty - \infty$, ∞/∞, $0 \times \infty$, and ∞^0

Rule: If $f(a) = g(a) = 0$ or
$f(x) \to \pm\infty$ and $g(x) \to \pm\infty$
as $x \to a$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

that is, direct plug in produces $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Ex. 1 $\frac{0}{0}$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{2x}{1} = 4$$

Ex. 2 $\frac{0}{0}$

$$(1+x)^{\frac{1}{2}} - 1 - \tfrac{1}{2}x$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} =$$

$\frac{0}{0}$ again

$$\lim_{x \to 0} \left[\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}} - \frac{1}{2}}{2x} \cdot \frac{2}{2} = \frac{(1+x)^{-\frac{1}{2}} - 1}{4x} \right]$$

$$\lim_{x \to 0} \left[\frac{-\frac{1}{2}(1+x)^{-3/2}}{4} = \frac{-(1+x)^{-3/2}}{8} \right] = -\frac{1}{8}$$

Find the following limits:

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} =$$

$\frac{0}{0}$ case

L'hôpital applies

$$\lim_{x\to 0} \frac{1-\cos x}{3x^2} =$$

$$\lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

no indeterminate form $\Rightarrow$

L'hôpital does not apply

$$\lim_{x\to 4} \frac{2x-7}{3x-11} = \frac{1}{1} = 1$$

$\frac{0}{0}$ L'hôpital applies

$$\lim_{x\to 0^-} \frac{\sin x}{x^2} = \lim_{x\to 0^-} \frac{\cos x}{2x}$$

$\frac{\oplus}{\ominus}$ $\frac{1}{0}$

$$= -\infty$$

$$\lim_{x\to\infty} \frac{2x^3 - x}{5 - x^3} =$$

$\frac{\infty}{\infty}$ L'hôpital applies

$$\lim_{x\to\infty} \frac{6x^2 - 1}{-3x^2} = \lim_{x\to\infty} \left[\frac{12x}{-6x} = -2\right]$$

$$= -2$$

∞/∞

$$\lim_{x\to\frac{\pi}{2}} \left[\frac{\tan x}{1 + \sec x} = \frac{\sin x / \cos x}{1 + \frac{1}{\cos x}} \cdot \frac{\cos x}{\cos x} = \frac{\sin x}{\cos x + 1}\right]$$

$$= \frac{1}{0+1} = \frac{1}{1} = 1$$

$$\lim_{x\to\infty} x \sin\frac{1}{x} \qquad \infty \cdot 0$$

let $u = \frac{1}{x}$

$x = \frac{1}{u}$

$$\lim_{u\to 0}\left[\frac{1}{u}\cdot \sin u = \frac{\sin u}{u}\right] =$$

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NORMAL FLOAT AUTO REAL RADIAN CL
999*sin(1/999)
                  .999999833
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$$\lim_{u\to 0}\frac{\cos u}{1} = 1$$

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \qquad \infty - \infty$$

$$\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) = \lim_{x\to 0}\frac{1-\cos x}{x\cos x + \sin x}$$

$$= \lim_{x\to 0}\frac{\sin x}{-x\sin x + \cos x + \cos x}$$

$$= \frac{0}{1+1} = \frac{0}{2} = 0$$

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}}$$ 1^∞

$$\boxed{y = \lim_{x\to 1^+} x^{\frac{1}{1-x}}}$$

$$\ln y = \lim_{x\to 1^+} \ln x^{\frac{1}{1-x}} = \lim_{x\to 1^+} \frac{1}{1-x}\cdot \ln x =$$

$$\lim_{x\to 1^+} \frac{\ln x}{1-x} \quad 0/0 \quad = \lim_{x\to 1^+} \frac{\frac{1}{x}}{-1} = \lim_{x\to 1^+} -\frac{1}{x} = -1$$

$$e^{\ln y} = e^{-1} \Rightarrow y = e^{-1} = \frac{1}{e}$$

$$\lim_{x\to 0^+} x^x$$

$$y = \lim_{x\to 0^+} x^x$$ 0^0

let $u = \frac{1}{x}$

$$\ln y = \lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x \ln x$$

$$= \lim_{u\to\infty} \frac{1}{u} \ln \frac{1}{u} = \lim_{u\to\infty} \frac{\ln u^{-1}}{u}$$

∞/∞

$$= \lim_{u\to\infty} \frac{-\ln u}{u} = \lim_{u\to\infty} \frac{-\frac{1}{u}}{1}$$

$$= 0$$

$$e^{\ln y} = e^{0} \Rightarrow y = e^0 = 1$$

$$\lim_{x\to\infty}\frac{\sqrt{5x+1}}{\sqrt{3x-5}}$$

$$= \lim_{x\to\infty}\sqrt{\frac{5x+1}{3x-5}}$$

$$= \sqrt{\lim_{x\to\infty}\frac{5x+1}{3x-5}}$$

$$= \sqrt{\lim_{x\to\infty}\frac{5}{3}} = \frac{\sqrt{5}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{15}}{3}$$